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A two-scale finite element formulation of Stokes flow in porous media

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ABSTRACT

Seepage through saturated porous material with an open pore system is modeled as a non-linear Stokes flow through a rigid matrix. Based on variationally consistent homogenization, the resulting macroscale problem becomes a Darcy-type flow. The prolongation of the Darcy flow fulfills a macrohomogeneity condition, which in a Galerkin context implies a symmetric macroscale problem. The homogenization is of 1st order and periodic boundary conditions are adopted on a Representative Volume Element. A nonlinear nested multiscale technique, in which the subscale problem is used as a constitutive model, is devised. In the presented numerical investigation, the effects of varying physical parameters as well as of the discretization are considered. In particular, it is shown that the two-scale results agree well with those of the fully resolved fine-scale problem.

1. Introduction

In this paper, we consider fluid flow through a porous material characterized by a solid skeleton that provides a substructure of a open pore system in which the fluid is contained. The substructure is generally very complex and has a characteristic scale much smaller than the size of the engineering component. In practice, it is virtually impossible to resolve the detailed flow characteristics due to both the large difference in size of the component and its microsctructure and the uncertainty in determining the complete spatial variation of the subscale features. In addition, numerous applications of engineering interest involve non-linearities on the subscale that result in a complex behavior on the macroscale. Examples are flow of moderate-to-high Reynolds numbers in the pore system, non-Newtonian and multi-phase flows through porous media, and the problem of truly coupled deformation and seepage. These obstacles are commonly tackled by the use of macroscopic models such as, e.g. Porous Media Theory, cf. [1]. However, non-linear phenomenological models on the macro-scale are difficult to construct and calibrate.

The most commonly adopted macroscale model for seepage in porous media is the so-called Darcy's law, which in its purest form is a linear, phenomenological, relation between the pressure gradient and the seepage velocity. This model of the permeability is combined with the continuity equation to give a boundary value problem in terms of the fluid pressure. The permeability model, e.g. the constitutive relation between fluid pressure (gradient) and seepage, can be established either by experimental measurements or via analysis, i.e. homogenization. Homogenization of porous media has been studied, for instance, using asymptotic expansion of the pertinent unknown functions. See e.g. [2–5] for periodic substructures and [6,7] for random substructures. A constant macroscale permeability tensor can be established via up scaling only for the special case of a completely linear flow on the pertinent subscale. Examples of work with such an upscaling include [8,9] from the field of Resin Transfer Molding and [10,11] from the field of oil geology.

The key step in homogenization is to establish a local boundary value problem on a Representative Volume Element (RVE) pertaining to the subscale, as shown in e.g. [12–16]. In other words, the macroscale response is directly given in terms of the solution of a subscale problem. The size of the RVE must be large compared to a characteristic length of the microstructure. In its simplest form, homogenization exploits a first order expansion of a macroscale quantity which is imposed on the subproblem by using the pertinent boundary conditions (commonly referred to as *prolongation*). Thus, in first order homogenization, the imposed macroscale quantity varies linearly on the RVE while in second order homogenization the imposed macroscale quantity varies quadratically. First and second orders of homogenization are discussed in e.g. [17]. To completely define the prolongation, different boundary conditions or loads on the RVE are used. It is important to note that the linear expansion introduces an error as does the choice of boundary conditions or loads.

The nested analysis required for the complete two-scale analysis of non-linear problems is commonly denoted FE^2, since the Finite Element Method (FEM) is used for the numerical solution on both the macro- and subscales, cf. [18]. Among the vast amount of literature on the topic, see e.g. [19–21]. A key ingredient in the

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procedure is the derivation of macro-scale tangent relations required in order to establish Newton iterations on the macroscale, cf. [22,23].

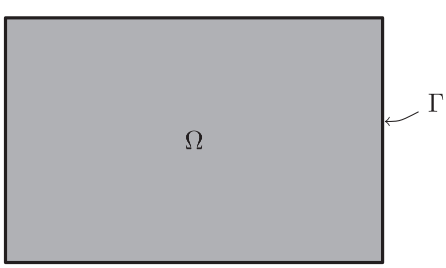
In this work, nonlinear and incompressible Stokes' flow through a rigid solid skeleton is considered, and we adopt the strategy proposed by Sandström and Larsson, cf. [24], for variationally consistent homogenization of Stokes flow in porous media. The nested procedure for the two-scale analysis is presented. In particular, the local sensitivity problems pertinent to a macro-scale tangent permeability tensor are defined. Examples are presented in 2D, where the homogenization procedure is illustrated for both linear and non-linear flows. As this work considers flow in a porous material, the fluid part of the RVE is required to consist of a topologically connected domain that allows for in- and out flow in all directions. For a subscale representation not fulfilling these requirements, the size of the RVE is either to small or the pore system is closed, resulting in a non permeable material. Seepage in an open pore system in 2D corresponds to flow around obstacles on the subscale.

The paper is organized as follows: In Section 2, the theory of the concurrent multiscale approach is given. In Section 2.1, some basic definitions are given, followed in Section 2.2 by the pertinent equations concerning the macroscale. In Section 2.3, the subscale problem is discussed along with the macroscale tangent problem. Two numerical examples are shown in Section 3; one is concerned with the comparison of linear and nonlinear subscale flows whereas the other is concerned with a comparison of a homogenized domain versus a fully resolved domain.

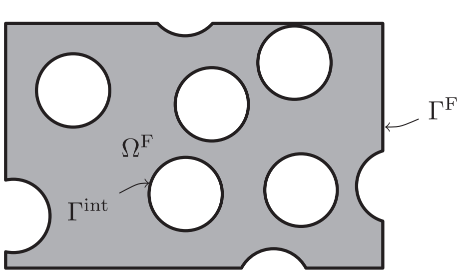
2. Concurrent multiscale method pertinent to a Stokes flow

2.1. Preliminaries

The homogenization starts out by considering a domain Ω consisting of the complete substructure of a porous material (Fig. 1). The pore domain $\Omega^{\mathrm{F}} \subset \Omega$ is filled with a fluid. Let Γ^{F} denote the part of $\partial\Omega^{\mathrm{F}}$ intersecting the boundary of Ω (cf. Fig. 1(b)), i.e. the part of $\Gamma := \partial\Omega$ where fluid can enter and exit the domain. The boundary Γ^{F} is further split into a Dirichlet part $\Gamma^{\mathrm{F}}_{\mathrm{V}}$ and a Neumann part $\Gamma^{\mathrm{F}}_{\mathrm{P}}$.



(a)



(b)

Fig. 1. (a) Example of a domain Ω containing a porous microstructure. (b) The fully resolved domain Ω where Ω^{F} denotes the porespace, Γ^{F} denotes the boundaries of the obstacles and Γ^{F} is the part of the bounary $\partial\Omega$ where fluid can enter and exit the domain.

At the internal boundaries, i.e. $\Gamma^{\mathrm{int}} \stackrel{\mathrm{def}}{=} \partial\Omega^{\mathrm{F}} \setminus \Gamma$, a no-slip condition is imposed.

The strong form of the fully resolved problem reads as follows:

$$-\nabla\cdot\boldsymbol{\sigma}(\boldsymbol{l},p) = \mathbf{0} \quad \text{in } \Omega^{\mathrm{F}} \tag{1a}$$

$$\nabla\cdot\boldsymbol{v} = 0 \quad \text{in } \Omega^{\mathrm{F}} \tag{1b}$$

$$\boldsymbol{t} \stackrel{\mathrm{def}}{=} \boldsymbol{\sigma}\cdot\boldsymbol{n} = -\hat{p}\boldsymbol{n} \quad \text{on } \Gamma^{\mathrm{F}}_{\mathrm{P}} \tag{1c}$$

$$\boldsymbol{v} = \hat{v}_n\boldsymbol{n} \quad \text{on } \Gamma^{\mathrm{F}}_{\mathrm{V}} \tag{1d}$$

$$\boldsymbol{v} = \mathbf{0} \quad \text{on } \Gamma^{\mathrm{int}} \tag{1e}$$

where $\boldsymbol{v}$ is the velocity, p is the pressure, $\boldsymbol{l} \stackrel{\mathrm{def}}{=} [\boldsymbol{v}\otimes\nabla]$ is the velocity gradient, $\boldsymbol{t}$ is the Cauchy traction, $\boldsymbol{n}$ is the outward pointing normal and $\boldsymbol{\sigma}$ is the Cauchy stress. Furthermore, $\hat{p}$ is the prescribed pressure and $\hat{v}_n$ is the prescribed velocity in the direction of $\boldsymbol{n}$. Subsequently, we shall adopt the formulation presented in [24] and introduce a two-scale formulation.

Remark 1. The boundary conditions in Eq. (1) are chosen to be of pressure-inflow type. This is done in order to comply with a Darcy-type seepage on the macroscale, cf. [24].

In the following discussion, the generally nonlinear relation

$$\boldsymbol{\sigma}(\boldsymbol{l},\ p) = \boldsymbol{\sigma}^{\mathrm{v}}(\boldsymbol{l}) - p\boldsymbol{I} \tag{2}$$

is adopted, where $\boldsymbol{\sigma}^{\mathrm{v}}$ is the viscous (deviatoric) stress tensor. In the absence of micropolar effects, which is the standard assumption adopted subsequently, $\boldsymbol{\sigma}^{\mathrm{v}}$ is symmetric and only influenced by the symmetric part of $\boldsymbol{l}$.

2.2. The macroscale problem

Assuming separation of scales, we consider the macroscale as the homogeneous domain Ω (cf. Fig. 1). In each macroscale coordinate $\bar{\boldsymbol{x}}$ we introduce a Representative Volume Element $\Omega_\square$ within which we assume the macroscale fields to vary linearly. Hence, we adopt first order homogenization.

We consider the macroscale problem, derived in [24], as follows:

$$\nabla\cdot\bar{\boldsymbol{w}}\{\bar{p},\nabla\bar{p}\} = 0 \quad \text{in } \Omega \tag{3a}$$

$$\bar{\boldsymbol{w}}\cdot\boldsymbol{n} = \hat{w} \quad \text{on } \Gamma_{\mathrm{V}} \tag{3b}$$

$$\bar{p} = \hat{p} \quad \text{on } \Gamma_{\mathrm{P}} \tag{3c}$$

where $\{\bullet\}$ implies implicit dependence and $\bar{\boldsymbol{w}}$ is the subscale response due to the macroscale pressure $\bar{p}$ and defines seepage (cf. Section 2.3) for the underlying subscale as

$$\bar{\boldsymbol{w}} \stackrel{\mathrm{def}}{=} \phi\langle\boldsymbol{v}\rangle_\square \tag{4}$$

which locally is determined by the macroscale pressure $\bar{p}$ and its gradient. In Eq. 4, the intrinsic average over the fluid domain $\Omega^{\mathrm{F}}_\square$ was introduced as

$$\langle f\rangle_\square = \frac{1}{|\Omega^{\mathrm{F}}_\square|}\int_{\Omega^{\mathrm{F}}_\square} f\,\mathrm{d}V \tag{5}$$

for an arbitrary function f, and the porosity is defined as

$$\phi = \frac{|\Omega^{\mathrm{F}}_\square|}{|\Omega_\square|} \tag{6}$$

Note that, although $\Gamma^{\mathrm{F}}_{\mathrm{V}} \subset \Gamma_{\mathrm{V}}$ represents a Dirichlet part of the boundary in the fully resolved problem, it pertains to a Neumann