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A recovery-explicit error estimator in energy norm for linear elasticity

E. Nadal[a,*], P. Díez[b], J.J. Ródenas[a], M. Tur[a], F.J. Fuenmayor[a]

[a] *Centro de Investigación de Tecnología de Vehículos (CITV), Universitat Politècnica de València, E-46022-Valencia, Spain*
[b] *LaCaN, Universitat Politècnica de Catalunya & CIMNE, Barcelona, Spain*

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Abstract

Significant research effort has been devoted to produce one-sided error estimates for Finite Element Analyses, in particular to provide upper bounds of the actual error. Typically, this has been achieved using residual-type estimates. One of the most popular and simpler (in terms of implementation) techniques used in commercial codes is the recovery-based error estimator. This technique produces accurate estimations of the exact error but is not designed to naturally produce upper bounds of the error in energy norm. Some attempts to remedy this situation provide bounds depending on unknown constants. Here, a new step towards obtaining error bounds from the recovery-based estimates is proposed. The idea is (1) to use a locally equilibrated recovery technique to obtain an accurate estimation of the exact error, (2) to add an explicit-type error bound of the lack of equilibrium of the recovered stresses in order to guarantee a bound of the actual error and (3) to efficiently and accurately evaluate the constants appearing in the bounding expressions, thus providing asymptotic bounds. The numerical tests with *h*-adaptive refinement process show that the bounding property holds even for coarse meshes, providing upper bounds in practical applications.

Keywords: Error bounding; Recovery techniques; Explicit residual error estimator

1. Introduction

Although the Finite Element Method (FEM) is a powerful method for a vast type of engineering problems, it is well known that, in general, it is only able to provide an approximated solution. Therefore, some *error* level has to be accounted for to define the safety factors during the design process of structural components. During several years various error estimation techniques have been developed. We can classify the error estimators in three groups based on its convergence through the global effectivity index θ (ratio of the estimate to the true error):

- *Asymptotically exact*: when the richness of the discrete solution space N is increased, the estimated error gets closer (from above or below, or even oscillating) to the true one, then $\theta \to 1$ when $\dim N \to \infty$.

* Corresponding author.
E-mail addresses: ennaso@upv.es (E. Nadal), pedro.diez@upc.edu (P. Díez), jjrodena@mcm.upv.es (J.J. Ródenas), matuva@mcm.upv.es (M. Tur), ffuenmay@mcm.upv.es (F.J. Fuenmayor).

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- *Asymptotically (upper) bounded*: when the richness of the discrete solution space is increased, the estimated error provides higher values than the true one, therefore $\theta \geq 1$ when $\dim N \to \infty$.
- *Asymptotically not bounded*: when the richness of the discrete solution space is increased, the estimated error provides lower values than the true one, then $\theta \leq 1$ when $\dim N \to \infty$.

Another way to classify the error estimators is according to the procedure used to obtain the estimates. Traditionally, there are three major branches in the error estimation field: The first group, the residual-based error estimators, introduced by Babuška and Rheinboldt [1] is subdivided into implicit [2–4] and explicit ones [5]. The explicit ones depend on a constant. Explicit values for the constant have been obtained by several authors [6,7]. More recently Prof. Stein and co-workers [8,9] have obtained an explicit value which provides highly accurate error estimations for a certain type of problems and elements. The second branch of error estimators, related with the concept of dual analysis, makes use of two solutions, one compatible and one equilibrated. Some of these error estimators solve two global problems in parallel [10] whereas other post-process the FE solution [11–13]. Under this group we can also include the error estimators based on the Constitutive Relation Error (CRE) introduced by Ladevèze and Leguillon [11] and followed by several contributions for many applications, see for example [14–17]. This technique compares a kinematically admissible stress field with a statically admissible solution obtained by solving local problems, built using the strong prolongation condition. These types of error estimators are *asymptotically (upper) bounded*. Finally, the third branch, the recovery-type error estimators are based on the use of the Zienkiewicz and Zhu (ZZ) error estimator [18]. These techniques were traditionally unable to provide upper bounds of the error in energy norm. The error evaluation is obtained by calculating the energy norm associated with the difference between the FE solution (compatible) and a recovered stress solution (not necessary equilibrated, but continuous), obtained for example with the Superconvergent Patch Recovery (SPR) technique [19,20]. In this case the recovered stress field is obtained processing the values of the stress field evaluated at superconvergent points. The process produces a recovered solution with higher convergence rate than the theoretical optimum convergence rate of the FE solution. This recovered solution, of a higher accuracy than the FE solution, is introduced in the ZZ error estimator providing accurate results. An intensive analysis of the superconvergent property for different recovery procedures can be found in [21]. Refs. [22–24] showed that under certain assumptions, related with the mesh type and regularity of the solution, when the recovered field used for the estimation is obtained with the SPR technique, the error estimator is *asymptotically exact*. The recovery-based error estimators are robust, easy to implement and are used in commercial codes. The publication of the original SPR technique was followed by several works aimed to improve its quality, see for example [25–27]. Ródenas et al. proposed to add constraints to impose local equilibrium and local compatibility to the recovered solution in the FEM framework [28] bringing up the SPR-C technique that was also adapted to the eXtended Finite Element Method (XFEM) framework [29,30].

Carstensen and Funken [31,32] presented an error estimator, based on recovery-type error estimators, providing upper bounds of the error in energy norm, under certain assumptions of smoothness of the solution which permit to neglect the lack of internal equilibrium of the recovered solution. Díez et al. [33] presented a methodology to obtain computable upper bounds of the error in the energy norm considering the stress recovered field provided by the SPR-C technique and taking into account the lack of internal equilibrium of the recovered stresses. This technique allowed to obtain one of the firsts procedures to get practical upper bounds for FEM and XFEM based on recovery techniques [33,34]. Since the SPR-C technique does not provide an equilibrated stress field at the global level, the upper bound property is obtained by adding a correction term to account for equilibrium defaults. So far, only a computationally expensive estimation, based on projection techniques, of these correction terms is available.

In this work we use the Cartesian Grid FEM (cgFEM) presented in [35–37] to numerically solve the linear elasticity problem, although, we have to remark that all results presented in this work can be directly extended to the standard FEM. The cgFEM is an immerse boundary method where the geometry is embedded into the mesh domain. Nested meshes are used in cgFEM for h-adaptive analysis. Therefore, projection techniques used in the proposed error estimation are efficient and easy-to-implement. In any case, the projections imply a low computational cost as they are only required in the first meshes of the h-refinement analysis. Some authors claim that the use of the SPR technique with Cartesian Grids, as the one used in this work, would be problematic. For instance, Ref. [38] indicates: "*Unfortunately, for an implicit mesh it would be very difficult to implement such a superconvergent recovery scheme of the stress field for elements that intersect the boundary*". However in the XFEM framework, where the mesh is independent of the crack, efficient recovery techniques have been already proposed based on the Moving Least Squares (MLS) technique [39–43] and some on the SPR technique [30,34], which introduce worthy improvements to the solution,