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A reduced basis method with exact-solution certificates for steady symmetric coercive equations

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Highlights

- Reduced basis method that provides rigorous bounds of the exact PDE energy.
- Upper bound computation is based on variational argument.
- Lower bound computation is based on complementary variational principle.
- Bound evaluation permits offline–online computational decomposition.

Abstract

We introduce a reduced basis method that computes rigorous upper and lower bounds of the energy associated with the infinite-dimensional weak solution of parametrized steady symmetric coercive partial differential equations with piecewise polynomial forcing and operators that admit decompositions that are affine in functions of parameters. The construction of the upper bound appeals to the standard primal variational argument; the construction of the lower bound appeals to the complementary variational principle. We identify algebraic conditions for the reduced basis approximation of the dual variable that results in an exact satisfaction of the dual feasibility conditions and hence a rigorous lower bound. The formulation permits an offline–online computational decomposition such that, in the online stage, the approximation and exact certificates can be evaluated in complexity independent of the underlying finite element discretization. We demonstrate the technique in two numerical examples: a one-dimensional reaction–diffusion problem with a parametrized diffusivity constant; a planar linear elasticity problem with a geometry deformation. We confirm in both cases that the method produces guaranteed upper and lower bounds of the energy at any parameter value, for any finite element discretization, and for any reduced basis approximation.

Keywords: Reduced basis; *a posteriori* error bound; Complementary variational principle; Partial differential equations

1. Introduction

The theory and applications of the certified reduced basis method – a model reduction technique that aims to achieve a rapid and reliable characterization of parametrized partial differential equations – have advanced considerably in the past decade (see Rozza et al. [1], Quarteroni et al. [2], and references therein). A computationally efficient offline–online construction of error bounds has been one of the main focuses of the certified reduced basis method;

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however, to our knowledge, with few recent exceptions [3,4], the existing reduced basis error bounds are with respect to some finite element "truth" which is *assumed* to be sufficiently accurate. This assumption may not be true for problems with spatial singularities and in any event is often not verified in a rigorous manner. The lack of reliable feedback on the validity of the "truth" can lead to either an inaccurate reduced basis prediction in the online stage (with respect to the infinite-dimensional weak solution) or overly conservative finite element "truth" and expensive computation in the offline stage. In this work, we introduce a reduced basis method that provides rigorous upper and lower bounds of the energy associated with the infinite-dimensional weak solution of parametrized steady symmetric coercive partial differential equations under two assumptions: the equation of interest consists of piecewise polynomial forcing functions (or "data"); the differential operator and data admit decompositions that are affine in functions of the parameters. We hence aim to entirely remove the issue of the "truth" within the certified reduced basis framework.

Our reduced basis method appeals to the complementary variational principle (or constitutive relation error), a principle that has been successfully used in the construction of finite element error bounds by, for instance, Ladevèze and Leguillion [5], Ainsworth and Oden [6], and Sauer-Budge et al. [7,8]. More recently, in the context of model reduction, the principle has been used in the construction of rigorous error bounds for the proper generalized decomposition by Ladevèze and Chamoin [9]; the principle has also been applied to computational homogenization by Kerfriden et al. [10]. In the context of certified reduced basis method for *parametrized* partial differential equations, the key to the application of the complementary variational principle is the identification of algebraic conditions for the dual reduced basis space associated with an *arbitrary parameter value*. In particular, the construction of the dual reduced basis space must be independent of the finite element complexity. We will demonstrate that this is indeed possible under the usual reduced basis assumption of the affine parameter dependence.

The contribution of the paper is the reduced basis formulation that provides upper and lower bounds for the energy associated with the infinite-dimensional weak solution of steady symmetric coercive equations. The bounds are uniform, as opposed to asymptotic, and certifies the approximation for any parameter value, for any finite element resolution, and for any reduced basis resolution. In addition, for our particular bound construction, we may associate the bound gap of the energy with the energy norm of the reduced basis solution error, and hence the energy bound provides an exact certificate of the solution field. The method admits an offline–online computational decomposition such that the online computational cost, including the cost associated with the bound computation, is independent of the underlying finite element discretization. As mentioned above, the formulation removes the issue of the finite element "truth" in the reduced basis certification process; the formulation is particularly suited for problems that exhibit spatial singularity in which the reliability of the finite element "truth" space can be questionable.

Before we proceed, we note limitations of the proposed bound strategy based on the complementary variational principle; the first three are inherited from the finite element counterpart. First, the method applies only to symmetric coercive problems. Second, the method requires the "data" – both from the interior forcing and boundary conditions – that is exactly representable as piecewise polynomials with respect to the underlying finite element triangulation. Third, the formulation requires, in the offline stage, a non-standard finite element approximation of the dual-feasible solution. Fourth, the formulation requires, in the online stage, a potentially large algebraic expansion, which could compromise the online efficiency. We discuss potential extensions that could remedy some of these limitations at the conclusion of this paper in Section 5.

This paper is organized as follows. In Section 2, we introduce our reduced basis method with exact-solution certificates for a diffusion equation with a parametrized diffusivity tensor; we recall the complementary variational principle, review a computational strategy for the dual variables, and present an offline–online computational strategy for the reduced basis method. In Section 3 we consider various extensions of the method: we consider a parametrized right-hand side, multiple domain-dependent parameters, reaction–diffusion equation, (planar) linear elasticity equations, and affine geometry transformations. In Section 4, we demonstrate the method on two examples: one-dimensional reaction–diffusion equation with a variable diffusivity constant; planar-stress linear elasticity with an affine geometry transformation. In Section 5, we summarize the key contributions and identify several future research directions.

2. A reduced basis method with exact-solution certificates

2.1. Model problem: parametrized diffusion equation

By way of preliminaries, we first introduce a Lipschitz domain $\Omega \subset \mathbb{R}^d$ with a boundary partition $\partial\Omega = \Gamma_D \cup \Gamma_N$ for Γ_D non-empty. We then introduce a Hilbert space $V \equiv \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\}$ over Ω endowed with an inner