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The Generalized Empirical Interpolation Method: Stability theory on Hilbert spaces with an application to the Stokes equation

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Abstract

The Generalized Empirical Interpolation Method (GEIM) is an extension first presented by Maday and Mula in Maday and Mula (2013) in 2013 of the classical empirical interpolation method (presented in 2004 by Barrault, Maday, Nguyen and Patera in Barrault et al. (2004)) where the evaluation at interpolating points is replaced by the more practical evaluation at interpolating continuous linear functionals on a class of Banach spaces. As outlined in Maday and Mula (2013), this allows to relax the continuity constraint in the target functions and expand both the application domain and the stability of the approach. In this paper, we present a thorough analysis of the concept of stability condition of the generalized interpolant (the Lebesgue constant) by relating it to an inf–sup problem in the case of Hilbert spaces. In the second part of the paper, it will be explained how GEIM can be employed to monitor in real time physical experiments by providing an online accurate approximation of the phenomenon that is computed by combining the acquisition of a minimal number, optimally placed, measurements from the processes with their mathematical models (parameter-dependent PDEs). This idea is illustrated through a parameter dependent Stokes problem in which it is shown that the pressure and velocity fields can efficiently be reconstructed with a relatively low-dimensional interpolation space.

Keywords: Empirical interpolation; Generalized empirical interpolation; Reduced basis; Model order reduction; Stability; Stokes equations

0. Introduction

Let $\mathcal{X}$ be a Banach space of functions defined over a domain $\overline{\Omega} \subset \mathbb{R}^d$ (or $\mathbb{C}^d$). Let $(X_n)_{n\in\mathbb{N}^*}$, $X_n \subset \mathcal{X}$, be a family of finite dimensional spaces, $\dim X_n = n$, and let $(S_n)_{n\in\mathbb{N}^*}$ be an associated family of sets of points: $S_n = \{x_i^n\}_{i=1}^n$,

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with $x_i^n \in \overline{\Omega}$. The problem of interpolating any function $f \in \mathcal{X}$ has traditionally been stated as:

$$\text{“Find } f_n \in X_n \text{ such that } f_n(x_i^n) = f(x_i^n), \quad \forall i \in \{1, \ldots, n\}\text{”,} \tag{1}$$

where we note that it is implicitly required that $\mathcal{X}$ is a Banach space of continuous functions. The most usual approximation in this sense is the Lagrangian interpolation, where the interpolating spaces X_n are of polynomial nature (spanned by plain polynomials, rational functions, Fourier series...) and the question on how to appropriately select the interpolating points in this case has broadly been explored. Although there exists still nowadays open issues on Lagrangian interpolation (see, e.g. [1]), it is also interesting to look for extensions of this procedure in which the interpolating spaces X_n are not necessarily of polynomial nature. The search for new interpolating spaces X_n is therefore linked with the question on how to optimally select the interpolating points in this case and how to obtain a process that is at least stable and close to the best approximation in some sense.

Although several procedures have been explored in this direction (we refer to [2,3] and also to the kriging studies in the stochastic community such as [4]), of particular interest for the present work is the Empirical Interpolation Method (EIM, [5–7]) that has been developed in the broad framework where the functions f to approximate belong to a compact set F of continuous functions ($\mathcal{X} = \mathcal{C}^0(\Omega)$). The structure of F is supposed to make any $f \in F$ be approximable by finite expansions of small size. This is quantified by the Kolmogorov n-width $d_n(F, \mathcal{X})$ of F in $\mathcal{X}$ (see definition (2)) whose smallness measures the extent to which F can be approximated by some finite dimensional space $X_n \subset \mathcal{X}$ of dimension n. Unfortunately, in general, the best approximation n-dimensional space is not known and, in this context, the Empirical Interpolation Method aims to build a family of interpolating spaces X_n with satisfactory approximation properties together with sets of interpolating points S_n such that the interpolation is well posed. This is done by a greedy algorithm on both the interpolating points and the interpolating selected functions φ_i (see [5]). This procedure has the main advantage of being constructive, i.e. the sequence of interpolating spaces (X_n) and interpolating points (S_n) are hierarchically defined and the procedure can easily be implemented by recursion.

A recent extension of this interpolation process consists in generalizing the evaluation at interpolating points by application of a class of interpolating continuous linear functionals chosen in a given dictionary $\Sigma \subset \mathcal{L}(\mathcal{X})$. This gives rise to the so-called Generalized Empirical Interpolation Method (GEIM). In this new framework, the particular case where the space $\mathcal{X} = L^2(\Omega)$ was first studied in [8]. We also mention the preliminary works of [9] in which the authors introduced the use of linear functionals in EIM in a finite dimensional framework. In the present paper, we will start by revisiting the foundations of the theory in order to show that GEIM holds for Banach spaces $\mathcal{X}$ (Section 1). The concept of stability condition (Lebesgue constant, Λ_n) of the generalized interpolant will also be introduced.

In the particular case where $\mathcal{X}$ is a Hilbert space, we will provide an interpretation of the generalized interpolant of a function as an oblique projection. This will shed some light in the understanding of GEIM from an approximation theory perspective (Section 2.1). This point of view will be the key to show that the Lebesgue constant is related to an inf–sup problem (Section 2.2) that can be easily computed (Section 3). The derived formula can be seen as an extension of the classical formula for Lagrangian interpolation to Hilbert spaces. It will also be shown that the Greedy algorithm aims to minimize the Lebesgue constant in a sense that will be made precise in Section 2.3. Furthermore, the inf–sup formula that will be introduced will explicitly show that there exists an interaction between the dictionary Σ of linear functionals and the Lebesgue constant. Although it has so far not been possible to derive a general theory about the impact of Σ on the behavior of the Lebesgue constant, we present in Section 4 a first simple example in which this influence is analyzed through numerical simulation.

The last part of the paper (Section 5) will allow to present some more elaborate potential applications of the method with respect to what is presented in [8]. In particular, we will explain how GEIM can be used to build a tool for the real-time monitoring of a physical or industrial process. This will be achieved thanks to the online computation of a generalized interpolant that will approximate the phenomenon under consideration. Its derivation will combine measurements collected on the fly from the process itself with a mathematical model (a parameter dependent PDE) that represents the physical understanding of the process. It will also be explained how the proposed methodology can be helpful for the minimization of the number of sensors required to reconstruct the field variable and also their optimal selection and placement, which are very important issues in engineering. These ideas will be illustrated through a parameter dependent Stokes problem for $\mathcal{X} = \left(H^1(\Omega)\right)^2 \times L^2_0(\Omega)$, where $L^2_0(\Omega)$ is the space of the $L^2(\Omega)$ functions with zero mean over Ω.