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Energy stable flux reconstruction schemes for advection–diffusion problems

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ABSTRACT

High-order methods for unstructured grids provide a promising option for solving challenging problems in computational fluid dynamics. Flux reconstruction (FR) is a framework which unifies a number of these high-order methods, such as the spectral difference (SD) and collocation-based nodal discontinuous Galerkin (DG) methods, allowing for their more concise and flexible implementation. Additionally, the FR approach can be used to facilitate development of new numerical methods that offer arbitrary orders of accuracy on unstructured grids. In previous work, it has been shown that a particular range of FR schemes, referred to as Vincent–Castonguay–Jameson–Huynh (VCJH) schemes, are guaranteed to be stable for linear advection problems for all orders of accuracy. There have remained questions, however, regarding the stability of FR schemes for advection–diffusion problems. In this study a new class of VCJH schemes is developed for solving one-dimensional advection–diffusion problems. For the first time, it is shown that the schemes are linearly stable for linear advection–diffusion problems for all orders of accuracy on nonuniform grids. Linear and nonlinear numerical experiments are performed in 1D and 2D to investigate the accuracy and stability properties of the new schemes. The results indicate that certain VCJH schemes for advection–diffusion problems possess significantly higher explicit time-step limits than discontinuous Galerkin schemes, while still maintaining the expected order of accuracy.

1. Introduction

Recent years have seen significant interest in the development of high-order methods for solving conservation laws. High-order methods produce less numerical dissipation relative to their lower order counterparts, allowing them to better resolve temporally evolving physical features. For example, in the context of fluid dynamics, high-order methods are known for their superior ability to preserve propagating vortex structures when simulating flows around rotorcraft, turbo-machinery, and flapping wings [1]. In addition, high-order methods perform more efficiently on problems with low error tolerances [2]. They have been successfully employed to simulate low-amplitude waves for applications in the areas of aeroacoustics and magnetohydrodynamics [3]. However, despite their advantages, high-order methods have yet to be adopted by the majority of fluid dynamicists. Instead, low order methods are more popular in industrial settings because they are more robust and easier to implement on the unstructured meshes which are frequently employed within complex geometries in

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practical applications. In order to remedy this situation, significant efforts have been devoted towards developing high order methods that are well-suited for unstructured grids.

The discontinuous Galerkin (DG) methods are perhaps the most well-known amongst high-order methods for unstructured grids. Traditional DG methods [4] have been successfully applied to the treatment of nonlinear conservation laws including the Euler and Navier–Stokes equations [5–7]. Recently, a nodal approach (referred to as collocation-based nodal DG) has gained popularity [2]. This approach is easier to implement than the (aforementioned) traditional DG methods as it is 'quadrature free' in the sense that it omits the explicit quadrature procedures associated with the traditional DG methods. In addition, the spectral difference (SD) approach, which was originally proposed in 1996 by Kopriva and Kolias [8], has gained popularity as a quadrature free method. This method was generalized in 2006 by Liu, Vinokur, and Wang [9], and has thereafter been successfully applied to a wide assortment of problems on unstructured grids [10–13].

Flux reconstruction (FR) emerged in 2007 as a single framework which encompasses a variety of collocation-based nodal DG and SD approaches. Originally proposed by Huynh [14], FR is a unifying framework for high-order methods, capable of recovering existing high-order schemes, and generating new schemes with favorable accuracy and stability properties. Using this framework, Huynh has identified FR schemes for advection–diffusion problems in 1D, and in 2D on quadrilaterals [15] and triangles [16]. In addition, Wang, Gao, Haga, and Yu have identified a closely related class of schemes referred to as Lifting Collocation Penalty (LCP) schemes [17–21] which have recently been extended to handle viscous terms [22]. In 2011, due to the similarity between the FR and LCP methods, the original developers of these methods decided to change their names to Correction Procedure via Reconstruction (CPR) [23,24]. Various CPR schemes have been successfully applied to solve the Navier–Stokes equations in 2D on triangular and quadrilateral elements [17] and in 3D on tetrahedral and prism elements [20,25].

Despite the success of 'FR-type' schemes (i.e. FR and LCP schemes), there remain questions regarding their stability for advection–diffusion problems. Thus far, efforts to prove the stability of the schemes have been incomplete. In particular, in [14,15] Huynh employed Fourier analysis to prove the stability of certain FR schemes. However, this analysis was restricted to uniform grids and a limited range for the order of accuracy. In response, a number of researchers have sought to prove the stability of FR schemes for all orders of accuracy on arbitrary (nonuniform) grids. In particular, in 2010 Jameson used an energy method to prove the stability of a particular SD scheme (an FR-type scheme) for 1D linear advection problems, for all orders of accuracy on nonuniform grids [26]. In numerical experiments, this 'energy stable' SD scheme was shown to exhibit a larger CFL limit than the collocation-based nodal DG scheme [27]. More recently, Vincent, Castonguay, and Jameson identified an entire class of FR schemes which they proved to be stable for linear advection problems in 1D (again for all orders of accuracy on nonuniform grids) [28]. These schemes, referred to as Vincent–Castonguay–Jameson–Huynh (VCJH) schemes [28], are parameterized by a single scalar c, and for particular choices of c, the SD scheme [26] and a collocation-based nodal DG scheme can be obtained for linear problems in 1D.

In this work, a new class of VCJH schemes is developed for solving *advection–diffusion* problems. It will be shown that the schemes are linearly stable on nonuniform grids for all orders of accuracy, thus demonstrating for the first time the stability of a class of FR schemes for advection–diffusion problems.

The format of the paper is as follows. Section two presents a FR approach for solving advection–diffusion problems in 1D. Section three introduces the VCJH correction functions. Section four develops a range of VCJH schemes for linear advection–diffusion problems, and uses an energy method to prove that these schemes are stable for all orders of accuracy. Finally, sections five and six present results of 1D linear and 2D nonlinear numerical experiments, with the aim of assessing how well the new VCJH schemes perform in practice.

2. Flux reconstruction approach for advection–diffusion problems

In this section, a new FR approach for advection–diffusion problems in 1D is presented. For readers unfamiliar with the FR approach for advection problems, the authors recommend a review of the procedure described in [14,28]. For readers unfamiliar with the FR approach for diffusion problems, the authors recommend a review of the procedure described in [15,29]. For the sake of completeness, the following procedure is written for the general class of advection–diffusion problems. In particular, the procedure presented here is applicable for general fluxes $f(u, \frac{\partial u}{\partial x})$ and utilizes distinct flux and solution correction functions.

2.1. Preliminaries

Consider the 1D conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f}{\partial x} = 0, \tag{1}$$

where x is the spatial coordinate, t is time, $u = u(x,t)$ is the conserved scalar quantity, and the flux $f = f(u, \frac{\partial u}{\partial x})$ is a nonlinear function of the solution u and its first spatial derivative. To eliminate terms involving second derivatives of the solution, Eq. (1) can be rewritten as a first-order system, as follows