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Consistency and convergence properties of the isogeometric collocation method

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ABSTRACT

Isogeometric collocation (IGA-C) method has shown its superior behavior over Galerkin method in terms of accuracy-to-computational-time ratio and other aspects. However, relatively little has been published about numerical analysis of the IGA-C method. This paper develops theoretical results on consistency and convergence of the IGA-C method to a generic boundary (initial) problem. It shows that the IGA-C method is convergent when differential operator of the boundary (initial) problem is stable or strongly monotone. Finally, we show some concrete examples whose differential operators are strongly monotone, and the IGA-C method is convergent. Moreover, 2D and 3D numerical examples are presented.

1. Introduction

Finite Element Analysis (FEA) gains widespread applications in physical simulation. However, while classical FEA methods are based on linear basis functions, CAD models are usually represented by NURBS with non-linear NURBS basis functions. When performing CAD model simulation, the NURBS-based CAD model should be transformed into linear mesh representation. As we all know, the operation of mesh transformation is very tedious, and it has become the most time-consumed task in the whole FEA procedure. Therefore, isogeometric analysis (IGA) is proposed by Hughes et. al. [1] to avoid the mesh transformation and to advance the seamless integration of CAD and CAE.

Since the IGA method is based on non-linear NURBS basis functions, it can deal with NURBS-based CAD models directly. And the IGA method can not only save lots of computation, but also greatly improve the computational precision. In addition, due to the knot insertion property of NURBS, the shape of CAD model can be exactly held in the refinement procedure [1]. Owning so many merits, the IGA method has been successfully applied in kinds of simulation problems, such as elasticity [2,3], structure [4–6], and fluid [7–9], etc.

For now, some work focuses on computational aspect of the IGA method and improves the accuracy and efficiency by using reparameterization and refinement, etc. [10–15]. Collocation method is a simple and efficient numerical method for solving differential equation, which can generate a numerical solution satisfying the differential equation at a set of discrete points, called collocation points [16]. If an unknown NURBS function is employed to approximate the analytical solution of a differential equation and its order is high enough, the collocation method can be applied to the strong form of the differential equation. Based on this fact, Auricchio et al. proposed the well-known isogeometric collocation (IGA-C) method [17]. For a boundary/initial problem with differential operator $\mathcal{D}$, we denote by T and T_r the analytic and numerical solutions,

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respectively, and n the number of the unknown coefficients of the NURBS function T_r. The IGA-C method first samples n values $\mathcal{D}T_r(\eta_i),\ i=1,2,\ldots,n$, and then generates a system of linear equations by $\mathcal{D}T_r$ interpolating these n values, i.e., $\mathcal{D}T(\eta_i)=\mathcal{D}T_r(\eta_i),\ i=1,2,\ldots,n$. The unknown coefficients of T_r can be determined by solving the linear system.

The IGA-C method has been extended to multi-patch NURBS configurations, various boundary and patch interface conditions, and explicit dynamic analysis [18]. Moreover, the IGA-C method has also been successfully employed in solving Timoshenko beam problem [19] and spatial Timoshenko rod problem [20], showing that mixed collocation schemes are locking-free independently of the choice of the polynomial degrees for the unknown fields. A comprehensive study on the IGA-C method reveals its superior behavior over Galerkin method in terms of accuracy-to-computational-time ratio [21]. Meanwhile, adaptive IGA-C methods are also developed and analyzed based on local hierarchical refinement of NURBS [21].

Unfortunately, a thorough numerical analysis of the IGA-C method is far from being established. Till now, all the analysis of the IGA-C method is only available for the one-dimensional case [17]. And the convergence results for 2D and 3D NURBS discretizations are available only based on numerical experiments [17,18].

In this paper, we present some theoretical consistency and convergence results of the IGA-C method for a generic differential operator. We first prove the consistency property of the IGA-C method. That is, for a PDE with the differential operator $\mathcal{D}$, T is its analytic solution, and a NURBS function T_r is the numerical solution. $\mathcal{D}T_r$ will tend to $\mathcal{D}T$, when each knot interval of T_r tends to a point. Then a theoretical convergence result is presented, and we prove that, if $\mathcal{D}$ is a stable or strongly monotone operator, the numerical solution T_r will tend to the analytic solution T when each knot interval of T_r tends to a point. Finally, we give some concrete examples where the differential operators are strongly monotone, and then the IGA-C method is convergent. It should be pointed out that, while the rate of convergence of the IGA-C method for one-dimensional problems is developed in [17], we just show the convergence of the IGA-C method for higher dimensional problems with stable or strongly monotone operator in this paper. Especially, when the differential operator is a stable or strongly monotone operator with polynomial coefficients, we present an error bound for the numerical solution generated by the IGA-C method.

The rest of this paper is laid out as follows. The generic formula of the IGA-C method is presented in Section 2. In Section 3, we study the knot vector of the derivative of arbitrary order of a NURBS function T_r, and prove that T_r and $\mathcal{D}T_r$ have the same breakpoint sequence and knot intervals. In Section 4, the theoretical consistency and convergence results of the IGA-C method are developed, and some concrete examples and numerical examples are presented. Finally, we conclude this paper in Section 5.

2. Generic formulation of the IGA-C method

A boundary value problem is expressed as

$$\begin{cases} \mathcal{D}T = f, & \text{in } \Omega \subset \mathbb{R}^d, \\ \mathcal{G}T = g, & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^d$ is a physical domain of d dimension, $\mathcal{D}$ is a bounded differential operator on the physical domain, $\mathcal{G}T = g$ is a boundary condition, and $f : \Omega \to \mathbb{R}$, $g : \partial\Omega \to \mathbb{R}$ are given functions. Suppose k is the maximum order of derivatives appearing in the operator $\mathcal{D} : V \to W$, where V and W are two Hilbert spaces, and the analytical solution $T \in C^m(\Omega),\ m \geqslant k$.

In the isogeometric analysis, the physical domain Ω is represented by a NURBS mapping:

$$\boldsymbol{F} : \Omega_0 \to \Omega, \tag{2}$$

where Ω_0 is a parametric domain. Replacing the control points of $\boldsymbol{F}(\Omega_0)$ by unknown control coefficients, we obtain the representation of numerical solution T_r, where $T_r \in C^k(\Omega)$.

Suppose there are n unknown control coefficients in the representation of T_r. We first sample n_1 points inside Ω_0, which correspond to n_1 values inside Ω, i.e., $\eta_i = \boldsymbol{F}(\theta_i),\ i=1,2,\ldots,n_1$. Next, we sample n_2 points on $\partial\Omega_0$, which correspond to n_2 values on $\partial\Omega$, i.e., $\eta_i = \boldsymbol{F}(\theta_i),\ i=n_1+1,n_1+2,\ldots,n_1+n_2$. The total number of theses points, called *collocation points*, should be equal to the number of the unknown coefficients of T_r, i.e., $n = n_1 + n_2$.

Inserting these collocation points into the boundary value problem (1) yields a system of equations with n equations and n unknowns, i.e.,

$$\begin{cases} \mathcal{D}T_r(\eta_i) = f(\eta_i), & i = 1,2,\ldots n_1, \\ \mathcal{G}T_r(\eta_i) = g(\eta_i), & i = n_1+1, n_1+2, \ldots, n. \end{cases} \tag{3}$$

Arranging the unknowns of T_r into an n-dimensional column vector, i.e., $X = [x_1\ x_2\ \cdots\ x_n]^{\mathrm{T}}$, the system of Eq. (3) can be represented in matrix form as

$$AX = b.$$

If the collocation points are so selected that the collocation matrix A is non-singular, the unknown coefficients X can be determined by solving the above mentioned system of linear equations.