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Eliminating spurious velocities with a stable approximation of viscous incompressible two-phase Stokes flow

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ABSTRACT

We present a parametric finite element approximation of two-phase flow. This free boundary problem is given by the Stokes equations in the two phases, which are coupled via jump conditions across the interface. Using a novel variational formulation for the interface evolution gives rise to a natural discretization of the mean curvature of the interface. In addition, the mesh quality of the parametric approximation of the interface does not deteriorate, in general, over time; and an equidistribution property can be shown for a semidiscrete continuous-in-time variant of our scheme in two space dimensions. Moreover, on using a simple XFEM pressure space enrichment, we obtain exact volume conservation for the two phase regions. Furthermore, our fully discrete finite element approximation can be shown to be unconditionally stable. We demonstrate the applicability of our method with some numerical results which, in particular, demonstrate that spurious velocities can be avoided in the classical test cases.

1. Introduction

It is well-known that non-physical velocities can appear in the numerical approximation of two-phase incompressible flows. These so-called spurious currents appear in different representations of the interface, with and without surface tension. If surface tension effects are taken into account, a jump discontinuity in the pressure results, and this poses serious challenges for the numerical method. As the pressure jump is balanced by terms involving the curvature of the unknown interface, it is necessary to accurately approximate the interface, its curvature and the pressure. In this paper we introduce a new stable parametric finite element method with good mesh properties, which leads to an approximation of the interface, and the conditions on it, with the property that undesired spurious velocities are either small or vanish completely. Although Bänsch, [3], proved a stability result for a (highly nonlinear) parametric discretization of the Navier–Stokes equations with a free capillary surface, to our knowledge, our stability result, for a linear scheme, is the first in the literature for a parametric discretization of two-phase flow.

When approximating two-phase flows, one has to decide first of all on how to represent the interface. The most direct choice is an explicit tracking of the interface. In these tracking methods the interface is either triangulated or represented by a connected set of particles, which carry forces. The interface is then transported using the flow velocity. Variants of these approaches are called front tracking methods, immersed interface methods or immersed boundary methods, see e.g. [39,27,3,30,19] for details.

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In a second completely different approach the interface is captured implicitly by defining a function on the whole domain. In the volume of fluid (VOF) method the characteristic function of one of the fluid phases is used in order to evolve the interface, see e.g. [24,33,31]. In the level set method, instead of a characteristic function, the interface is represented as the zero level set of a smooth function, see e.g. [37,36,29,22] for this approach. Finally, in the phase field approach, instead of a sharp interface description, the interface is considered to be diffuse with a small interfacial layer in which a phase field variable rapidly changes from the different constant values in the two phases, see e.g. [1,25,16].

Spurious velocities have already been observed in the numerical approximation of one-phase incompressible fluid flow with external forces, see [20], where a projection method to deal with this phenomenon is also proposed. In two-phase flow with surface tension effects it is well-known that the imbalance between the discrete computation of the curvature of the interface and the pressure jump at the interface can create spurious velocity fields near the interface, even in situations where the exact solution has zero velocity. Several methods to compute the discrete curvature and different choices of enriching the pressure space have been introduced to reduce spurious velocities, see e.g. [32,33,26,17,38,18,23,41,2].

In this paper we propose a numerical method for two-phase incompressible Stokes flow based on a parametric representation of the interface. We use finite elements to approximate the velocity and the pressure, and the interface is approximated using a lower dimensional mesh. Here we allow both for a fitted approach, where the bulk mesh is adapted to the interface, and an unfitted approach, where the bulk and interface meshes are totally independent. Typical meshes for both approaches are shown in Fig. 1. In this paper we use the unfitted approach for our numerical results, which means that we can avoid remeshings of the bulk mesh at every time step. As discussed for example in [18], unfitted bulk meshes for two-phase flow with pressure jumps, due to surface tension effects, lead to a poor approximation of the pressure. We avoid this by using locally refined bulk meshes at the interface in practice. But we stress that all our theoretical results presented in this paper also hold for fitted meshes. Another strategy in the context of the unfitted approach, which can be combined with our proposed method, is to enrich the pressure finite element space by functions providing additional degrees of freedom close to the interface; and, in particular, allow for pressure jumps in the elements cut by the interface. Such an enrichment technique is an example of the extended finite element method (XFEM), and has been used for two-phase problems in the context of level set methods. A major drawback of this approach is that the resulting linear algebraic system is typically very ill-conditioned, because the linear independence of the finite element basis deteriorates as enrichment functions with very small support arise. Although strategies have been developed to reduce the problem of ill-conditioning, see e.g. [23,2,34], the computational effort still remains large due to the reconstruction of the XFEM basis as the interface moves. An XFEM approach is also possible within the context of our method, and will be discussed later.

A common problem in approaches which directly parameterize an evolving interface is that typically the mesh deteriorates, and often computations cannot be continued without remeshing the interface. Situations often occur in which distances between some interface mesh points or some angles in the interface triangulation become very small. In earlier work, the present authors introduced a new methodology to approximate curvature driven curve and surface evolution, see [5,4,6]. The method has the important feature that interface mesh properties remain good during the evolution. In fact, for curves semidiscrete versions of the approach lead to polygonal approximations where the vertices are equally spaced throughout the evolution. The approach has been successfully used for various geometric evolution equations, including surface diffusion, [5], and grain boundary motion, [8], and was recently applied to crystal growth phenomena, see [9,10]. In this paper we generalize the approach to two-phase incompressible Stokes flow. The generalization to the Navier–Stokes case will be considered in the forthcoming article [12].

Studies of other groups reveal that the main source of spurious velocities in two-phase flow with surface tension is the fact that discontinuous functions allowing for jumps at the interface are not in the pressure space for an unfitted bulk mesh. In addition, it has been observed that the size of the spurious velocities depend both on the calculation of the interface

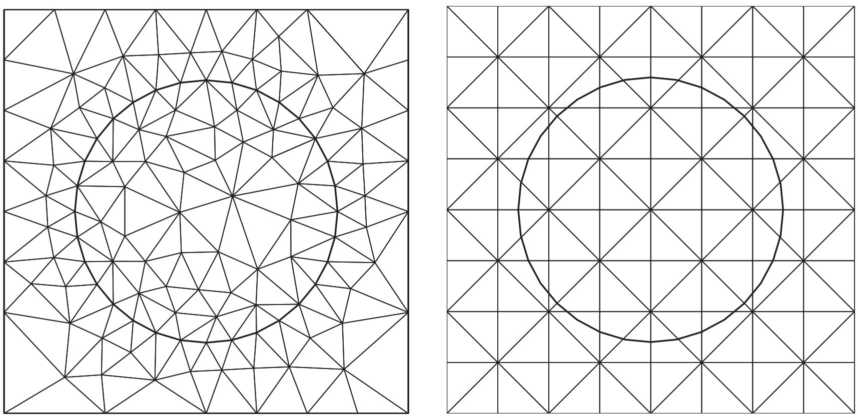

Fig. 1. Fitted and unfitted bulk finite element meshes for a circular interface.