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Adaptive *hpq* finite element methods for the analysis of 3D-based models of complex structures. Part 2. A posteriori error estimation

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ABSTRACT

This is the second paper out of a series of papers devoted to model- and *hpq*-adaptive finite element methods assigned for the modeling and analysis of elastic structures of complex mechanical description. In our previous publication we investigated the issue of hierarchical models and approximations of such structures. We applied 3D or 3D-based mechanical models, hierarchical modeling, and hierarchical approximations within the proposed finite element formulation. Furthermore, we assumed that the mechanical model and discretization parameters (such as: the size h of the element, and the longitudinal and transverse approximation orders, p and q) could vary locally, i.e. they could be different in each finite element. The a posteriori error estimation discussed in the present paper is based on the generalization of the residual equilibration method on the models with internal constraints. The generalized method is applied to the assessment of the total and approximation errors, while the modeling error is calculated as the difference between the former two errors. The corresponding error-controlled adaptive procedures are based on a three-step strategy, with possible iterations of h- and p-steps.

1. Introduction

1.1. The state-of-the-art issues

The main objective of this survey is to refer to the error estimation methods which we apply in this paper to the analysis of complex structures. We focus on the implicit residual methods, and the equilibrated residual method in particular.

1.1.1. A posteriori error estimation with implicit residual methods

Approximation error estimation. The implicit residual methods consist in the calculation of the element residua as measures of non-satisfaction of the element equilibrium equations and boundary conditions. This calculation takes advantage of the solutions to the local element problems. The obtained residua are the basis for the global error calculations in a chosen error norm. The most popular norms are the strain energy and L^2 ones. The main advantage of these methods is the upper bound property of the estimate of the global approximation error.

The residual methods of error estimation were introduced by Babuška and Rheinboldt [13] in 1978 for the 1D case, and then generalized for the 2D case [14] in the form of element residual methods (ERM). Further development of these methods

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can be attributed to Babuška, Zienkiewicz and their co-workers [28,26]. An additional contribution to these methods was made by Demkowicz et al. [24,25]. A great improvement of this approach was delivered by the works of Ainsworth and Oden [6,5], where a mature form of the residual equilibrated method (REM) was proposed. A different way of defining residua, based on posterior equilibrium method (PEM), was proposed by Stein and Ohnimus [48].

Ending, we would like to notice that the presented global residual methods of error estimation, based on the local solutions, may be incapable of detection of the pollution errors. Such errors are produced by the solution singularities at the points far from the elements under consideration, unless the strain energy is equidistributed in the mesh (see [16]).

Modeling error estimation. All of the above methods were primarily designed for the approximation error estimation. In some analyses, however, the total error is the sum of the approximation error (due to FE discretization of the solution domain) and the modeling error (due to the application of the physical model simplified with respect to the best available model of the phenomenon). In the case of mechanical problems, the examples of such simplified models can be the homogenized models applied to heterogeneous materials, or the dimensionally reduced models in solid mechanics.

The general considerations concerning the modeling error and its estimation were presented in the papers by Oden and Prudhomme [40,37]. They analyzed also the estimation of the modeling error in the multi-scale problems (see for example [42,39]). In relation to the classical solid mechanics problems, we should mention the initiating works by Babuška and Schwab [12,45,15], who analyzed hierarchic models of thin plates and shells. Their modeling error analyses were performed for the semi-discrete hierarchical modeling (the approximation error is negligible). The generalization of their approach to the case of fully discrete (the approximation error cannot be neglected) hierarchic models of thin structures has been proposed by Ainsworth [1]. In this approach, the total error is a sum of the modeling and approximation error components. The latter component is obtained with the residual equilibrated methods. Also, investigations of Oden and Cho [36,19], based on the residual equilibrated method, and concerning the plate- and shell-like structures, should be noted in this context. In their approach, the modeling error estimation is the result of the preceding total error estimation and some additional considerations concerning the a priori error estimation. Additionally, we can mention the application of the residual modeling error estimation within the structures of complex mechanical description, though without any proof [60,52]. In these propositions the modeling error estimation requires the total and approximation error estimation to be performed first. An example of the modeling error estimation, in the case of thin or thick first-order shells, performed with the PEM (posterior equilibrium method), was introduced and developed in works by Stein and Ohnimus [47,46].

The related comments on alternative error estimation methods. The residual methods described above are primarily designed for the estimation of the global errors and the element contributions to such errors. The alternative are the methods of point-wise error estimation (see [33,31,50] for the bibliography). The point-wise error estimation is the base for the so-called quantity of interest approach to error estimation. This approach, and the associated goal-oriented adaptivity, are sources of wide interest (see [43,38,41,18], for example). This interest results from the ability to cope with pollution errors – the feature not present in the case of the global estimators. In spite of this useful feature, we will not apply this approach, as our main goal is to demonstrate that there is some room for improvement in the case of the global residual estimators.

1.1.2. The residual equilibrated method

The beginnings and the following development of the residual equilibrated approach, applied to the approximation error estimation, can be found in the works of Ladevéze and Leguillon [32], Kelly [27], Bank and Weisser [17], and Ainsworth and Oden [5]. The application of this approach to the approximation error estimation in the finite element methods can be found in the works of Ainsworth and Oden [6,7]. The same authors, in [8,9,11], applied this method to the general elliptic and specific elasticity problems, respectively. As shown in the work by Oden and Cho [36,19], the residual equilibrated approach can also be applied to conventional (based on mid-surface dofs) hierarchical shell models. The recent advances in the development of this method include: its stability analysis [4], the generalizations to singularly perturbed reaction–diffusion problems [3], as well as conforming, non-conforming and discontinuous Galerkin finite element methods [2].

In most of the mentioned works, their authors started with an introduction of the ad hoc, residuum-loaded error functional, leading to the approximation error estimator, defined as the strain energy of the difference between the exact and the approximated solutions. An alternative was proposed by Oden and Cho [35], who started with the equivalent functional, based on the difference of the potential energies corresponding to the exact and the approximated solutions. They applied this approach to the total error estimation. In our previous paper [53], we applied the same approach to the estimation of the approximation error of the 3D-based first order shell model. In these approaches (followed also in this paper) one searches for the estimate of the exact solution, rather than for the estimated value of the approximation error. The error estimate is calculated next, as a difference between the solutions. The related implementation aspects were presented in [36,54].

1.2. Our definition of complex structures

In this paper complex structures are understood as elastic bodies described by more than one mechanical model, regardless of their geometrical complexity. This means that a simple plate can be treated as a complex one if various models, e.g. the first-order shell, transition and three-dimensional ones, are applied for the structure's mechanical characteristics. Conversely, a structure consisting of different geometrical parts (thin-walled, thick-walled and solid ones) can be treated as a simple one, if only one mechanical model (three-dimensional elasticity model, for example) is applied in its analysis. Summing up, it is the mechanical (or model) complexity that matters in this paper, while the geometrical one is irrelevant.