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Shishkin mesh simulation: A new stabilization technique for convection–diffusion problems ☆

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ABSTRACT

A new stabilization procedure is presented. It is based on a simulation of the interaction between the coarse and fine parts of a Shishkin mesh, but can be applied on coarse and irregular meshes and on domains with nontrivial geometries. The technique, which does not require adjusting any parameter, can be applied to different stabilized and non stabilized methods. Numerical experiments show it to obtain oscillation-free approximations on problems with boundary and internal layers, on uniform and nonuniform meshes and on domains with curved boundaries.

1. Introduction

The numerical solution of convection–diffusion problems when convection dominates is, despite more than 30 years of research, a challenging problem nowadays. Standard finite-element or finite-difference methods typically suffer from unphysical or spurious oscillations unless meshes are taken so fine that are useless for all practical purposes. The reason is the presence of layers or thin regions where solutions change fast. Modification of standard methods, known as stabilized methods have been proposed in the literature, from upwind methods 35 years ago [44], to strongly-consistent stabilized methods like the streamline upwind/Petrov–Galerkin (SUPG) method [7], also known as the streamline diffusion finite element method (SDFEM), or the Galerkin least-squares (GALS) method [23]. More recently, local projection stabilization (LPS) methods, [4,6,17], continuous interior penalty (CIP) methods [8], or discontinuous Galerkin (DG) methods [22,34] have been introduced, to cite a few of the many techniques proposed (see [35,37] for a survey of methods). It must be noticed, however, that computational studies (see e.g., [2,25]) find it hard to put a particular method above the others. It must be also mentioned that most of these methods depend on at least one parameter about which there is no unanimous agreement on its optimal choice in practical problems [26].

A different approach is to use layer-adapted meshes. Among these we cite Shishkin meshes (described below) [30,36], which have received considerable attention in recent years [13–15,27,31,32,38,42,46]. However, it is generally acknowledged that the main drawback of Shishkin meshes is the difficulty to design them on domains with nontrivial geometries, although some works overcoming this difficulty can be found in the literature [45,27].

The method we propose, however, does not suffer from the above indicated drawbacks: It does not depend on parameters and, although it is based on the idea of simulating a Shishkin mesh, the experiments we present show it produces excellent results on domains with nontrivial geometries.

We consider the problem

$$-\varepsilon\Delta u + b\cdot\nabla u + cu = f, \quad \text{in } \Omega, \tag{1}$$

$$u = g_1, \quad \text{in } \partial\Omega_D, \qquad \frac{\partial u}{\partial n} = g_2, \quad \text{in } \partial\Omega_N. \tag{2}$$

Here, Ω is a bounded domain in $\mathbb{R}^d$, $d = 1, 2, 3$, its boundary $\partial\Omega$ being the disjoint union of Γ_D and Γ_N, b and c are given functions and $\varepsilon > 0$ is a constant diffusion coefficient. We assume that $\Gamma^- \subset \partial\Omega_D$, Γ^- being the inflow boundary of $\Omega \subset \mathbb{R}^d$, i.e., the set of points $x \in \partial\Omega$ such that $b(x)\cdot n(x) < 0$.

It is well-known if $\varepsilon \ll \sup\{|b(x)||x \in \Omega\}$ ($|\cdot|$ being the euclidean norm) boundary layers are likely to develop along $\partial\Omega \setminus \Gamma^-$, although they have different structure on $\Gamma^0 = \{x \in \partial\Omega | b(x)\cdot n(x) = 0\}$ and $\Gamma^+ = \{x \in \partial\Omega | b(x)\cdot n(x) > 0\}$. As already mentioned, these boundary layers, when present, are responsible of the spurious oscillations that pollute the numerical approximations obtained with standard methods unless extremely fine meshes are used. For uniform meshes, oscillations typically disappear when the *mesh Péclet number*

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$$\mathrm{Pe} = \frac{\|b\|_{L^\infty(\Omega)^2} h}{2\varepsilon}$$

(h being the mesh size) is of the order of 1.

Let us briefly describe now the idea of the method we propose in the following simple problem:

$$L(u) \equiv -\varepsilon u''(x) + b(x)u'(x) + c(x) = f(x), \quad 0 < x < 1, \tag{3}$$
$$u(0) = u(1) = 0. \tag{4}$$

In (3) we assume that b, c and f are sufficiently smooth functions, and that

$$0 < \beta < \min_{x\in[0,1]} b(x), \quad 0 \leqslant \min_{x\in[0,1]} c(x). \tag{5}$$

The standard Galerkin linear finite-element method for (3) and (4) on a partition or mesh $0 = x_0 < x_1 < \cdots < x_J = 1$ of $[0,1]$ obtains a continuous piecewise linear approximation $u_h(x)$ to u. As it is customary, h denotes the mesh diameter, $h = \max_{1\leqslant j\leqslant J} h_j$, where $h_j = x_j - x_{j-1}$, for $j = 1,\ldots,J$. The approximation can be expressed as $U(x) = u_1\varphi_1(x) + \cdots + u_{J-1}\varphi_{J-1}(x)$, where the $\varphi_j(x)$ are the basis or hat (piecewise linear) functions taking value 1 at the node x_j and 0 in the rest of the nodes of the partition (thus, $U(x_j) = u_j$). The values u_j, $j = 1,\ldots,J-1$, are obtained by solving the linear system of equations

$$a(u_h, \varphi_i) = (f, \varphi_i), \quad i = 1,\ldots,J-1, \tag{6}$$

where, a is the bilinear form associated with (3), which is given by

$$a(v,w) = \varepsilon(v', w') + (bv' + c, w),$$

$(\cdot,\cdot)$ being the standard inner product in $L^2(0,1)$,

$$(f,g) = \int_0^1 f(x)g(x)\,\mathrm{d}x.$$

The Shishkin mesh with $J = 2N$ nodes is composed of two uniform meshes with N subintervals on each side of the transition point $x_N = 1 - \sigma$, where

$$\sigma = \min\left(\frac{1}{2}, \frac{2}{\beta}\varepsilon\log N\right),$$

for an adequate constant β, that is, $x_j = j(1-\sigma)/N$, for $j = 0,\ldots,N$, and $x_{N+j} = x_N + j\sigma/N$, for $j = 1,\ldots,N$. Let us consider the coarse and fine grid parts of the Galerkin approximation given by

$$U_c(x) = u_1\varphi_1(x) + \cdots + u_{N-1}\varphi_{N-1}(x),$$
$$U_f(x) = u_{N+1}\varphi_{N+1}(x) + \cdots + u_{2N-1}\varphi_{2N-1}(x),$$

so that $U_c + u_N\varphi_N + U_f$ is the Galerkin approximation on the Shishkin mesh. Since for $i = 1,\ldots,J-1$, the support of the basis function φ_i is $[x_{i-1}, x_{i+1}]$, we have $a(U_c, \varphi_{N+j}) = 0$ and $a(U_f, \varphi_j) = 0$, for $j = 1,\ldots,N-1$. Consequently the system (6) on the Shishkin mesh can be rewritten as

$$a(U_c, \varphi_i), = (f, \varphi_i), \quad i = 1,\ldots,N-2, \tag{7}$$
$$a(U_c, \varphi_i) + u_N a(\varphi_N, \varphi_i) = (f, \varphi_i), \quad i = N-1, \tag{8}$$
$$a(U_c, \varphi_N) + u_N a(\varphi_N, \varphi_N) + a(U_f, \varphi_N) = (f, \varphi_N), \tag{9}$$
$$u_N a(\varphi_N, \varphi_i) + a(U_f, \varphi_i) = (f, \varphi_i), \quad i = N+1,\ldots,2N-1. \tag{10}$$

We notice that were it not for the presence of the $u_N a(\varphi_N, \varphi_i)$ in (8), the system (7) and (8) would be the equations

$$a(U, \varphi_i) = (f, \varphi_i), \quad i = 1,\ldots,N-1, \tag{11}$$

of the Galerkin approximation $U = U_1\varphi_1 + \cdots + U_{N-1}\varphi_{N-1}$ for the problem

$$-\varepsilon u''(x) + b(x)u'(x) + c(x) = f(x), \quad 0 < x < 1-\sigma, \tag{12}$$
$$u(0) = u(1-\sigma) = 0. \tag{13}$$

The Galerkin approximation U for this problem, unless $\varepsilon N > 1/2$, is likely to have spurious oscillations of large amplitude as we show in Fig. 1 for $\varepsilon = 10^{-8}$, $\sigma = 4\varepsilon\log(J)$, $b(x) = f(x) = 1$, $c = 0$ and $N = 9$. It is however the presence of $u_N a(\varphi_N, \varphi_i)$ in Eq. (8) that suppresses the oscillations, as we can see in Fig. 1, where the component U_c of the Galerkin approximation on a Shishkin grid with $J = 2N = 18$ is also shown (discontinuous line) together with the true solution at the nodes of the coarse part of the mesh.

It is remarkable that just by adding the value

$$\alpha^* = u_N a(\varphi_{N-1}, \varphi_N) \tag{14}$$

to the last equation of the Galerkin method for (12) and (13) we get the oscillation-free approximation U_c. Obviously, in order to have the value of α^* we have to solve the whole system (7)–(10). In the present paper, we introduce a technique to approximate α^* without the need to compute the whole approximation on the Shishkin grid. In Fig. 1, the approximation computed with the estimated α^* is indistinguishable from U_c. Numerical experiments in the present paper show that, in two-dimensional problems, the oscillation-free approximation on a coarse mesh can be obtained by this technique at half the computational cost of a Shishkin grid, and a more substantial gain can be expected in three-dimensional problems.

Furthermore, this technique can be extended when the grid is no part of any Shishkin grid, while, at the same time, managing to get rid of the spurious oscillations. This allows to obtain accurate approximations on domains with non trivial boundaries, where Shishkin meshes may be difficult to construct. In spite of this, we call the new technique *Shishkin mesh simulation* (SMS), since it was derived, as described above, in an attempt to simulate Shishkin grids.

We must mention, however, that in the present paper we only consider the case of dominant convection, both in the analysis and in the numerical experiments. The question of how to modify the method (if necessary) when the mesh Péclet number Pe tends to one will be addressed elsewhere.

It is well-known that the Galerkin method is a far from ideal method in convection–diffusion problems. Let us also notice that despite the good properties of stabilized methods developed in recent years, the SUPG method is still considered the standard approach [26]. For this reason, in the numerical experiments presented below, we compare the new method with the SUPG method.

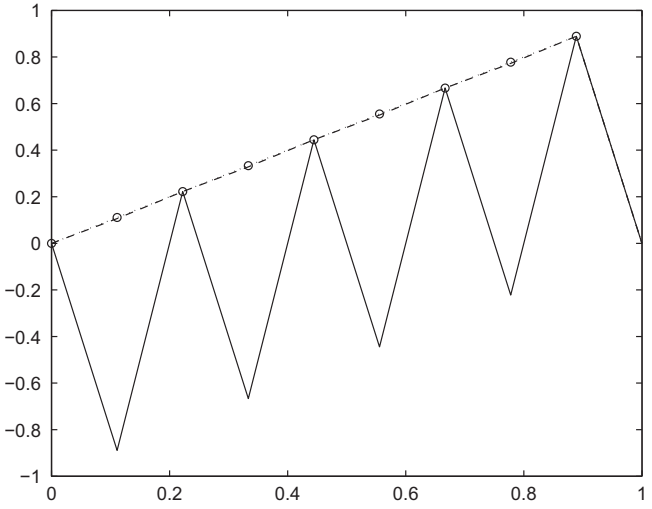

Fig. 1. Galerkin approximation on a uniform mesh with $N = 9$ (continuous line) to the solution of (12) and (13) with $\varepsilon = 10^{-8}$, $\sigma = 4\varepsilon\log(2*N)$, $b = f = 1$, and $c = 0$. The U_c part of the Galerkin approximation on a Shishkin mesh with $J = 18$ (broken line) for same ε and f. Circles are the values of the true solution on the nodes.