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Variable resolution for SPH: A dynamic particle coalescing and splitting scheme

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ABSTRACT

In this paper a novel variable resolution method using particle splitting and coalescing for the SPH numerical solution of the Navier–Stokes equations is presented. The key idea of the scheme is to modify dynamically the particle sizes by means of splitting and coalescing (merging) individual particles to provide good resolution only where it is needed. The SPH scheme adopted is derived using the variational principle guaranteeing that both mass and momentum are conserved for particles with different smoothing lengths. A particle shifting procedure is used to prevent unacceptable anisotropic distributions of the particles and is further generalized for treating domains with variable mass particles. The algorithm has been tested against analytical solutions for Poiseuille and Taylor–Green flows showing that the shifting algorithm is effective in increasing the accuracy, and that error introduced by the splitting and coalescing is negligible. The capability of the numerical scheme for increasing efficiency is shown for more general problems: the simulations of a moving square in a box and flow past a cylinder have shown that the particle refinement procedure is able to increase the efficiency while maintaining the same level of accuracy as a uniform distribution with the most refined resolution.

1. Introduction

In classical Eulerian computational models, adaptive structured [31] or unstructured grids [19] have been used successfully to provide variable resolution and to simulate multiscale flows while retaining computational efficiency. In meshfree numerical schemes there have been some early attempts to introduce variable resolution by either remeshing, and particle insertion/removal techniques [3,4,34,17]. Koumoutsakos [16] analysed the capability of the vortex method to simulate multiscale flows. Recently within the SPH formalism, dynamic particle refinement which conserves mass and momentum has been applied to the shallow water equations [37,35,36]. This has been obtained by particle splitting and coalescing procedures which can increase and decrease the spatial resolution and enabling simulations of practical large-scale flooding problems.

The dynamic particle refinement algorithm is developed in this work for the SPH solution of Navier–Stokes equations introducing a novel algorithm for merging or coalescing small particles into a larger particle. Since this procedure requires particles with different sizes, a consistent SPH discretization scheme which can discretize accurately the Navier–Stokes equations in the presence of variable smoothing length is necessary. Bonet and Rodríguez-Paz [2] proposed a momentum-conservative weakly compressible formulation which takes into account variable smoothing length. However, this is neither accurate in the presence of a free surface nor computationally efficient due to multiple sub-iterations required.

A variationally consistent and efficient SPH formulation is herein derived to address both of these issues. This scheme assures momentum conservation in presence of particles with different smoothing length, and moreover it addresses the efficiency and accuracy problem highlighted in [2].

Some additional improvement to the formulation in [2] is also introduced to increase the accuracy of the scheme. In the framework of projection-based incompressible SPH schemes Xu et al. [38] proposed a particle shifting algorithm which prevents instabilities due to highly disordered particle distributions. In this work the algorithm is generalized for particles with different masses.

This paper is organized as follows: in Section 2 the variationally consistent SPH discretization of Navier–Stokes equations is briefly reported and the particle shifting algorithm is also described, the complete derivation of the formulation is presented in Appendix A. In Section 3 the derivation of the particle splitting and coalescing algorithm is presented. In Section 4 the numerical scheme for Navier–Stokes equations is tested against analytical solutions of

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Pouseuille and Taylor–Green flow, a moving square in a box and finally the classical flow past a cylinder at Reynold numbers $Re = 20$ and $Re = 100$.

2. Variable resolution SPH algorithm

In this work we follow the classical SPH approach of weak compressibility usually adopted in SPH numerical schemes. Mass and momentum conservation for a pseudo-compressible fluid, can be written in Lagrangian form as:

$$\frac{d\rho}{dt} = -\rho\nabla\cdot\mathbf{v} \tag{1}$$

$$\frac{d\mathbf{v}}{dt} = +g - \frac{1}{\rho}\nabla p + v_0\nabla^2\mathbf{v} \tag{2}$$

where ρ is the fluid density, $\mathbf{v}$ is the velocity vector, p is the pressure, g is the gravity acceleration and v_0 is the kinematic viscosity.

The two equations are coupled by means of the equation of state:

$$p = B\left[\left(\frac{\rho}{\rho_0}\right)^{\gamma} - 1\right] \tag{3}$$

where γ is a constant taken equal to 7 as suggested by different authors [26,13], ρ_0 is the reference density and $B = c_0^2\rho_0/\gamma$ where c_0 is the speed of sound in the fluid. The speed of sound is conveniently reduced to obtain a larger computational timestep. In this work the speed of sound is defined as $c_0 = 10U_{max}$ where U_{max} is the maximum expected fluid velocity.

We remark that the smoothing length of a single particle does not change in time, but different smoothing lengths are created by the particle splitting and coalescing procedure presented herein, and hence, must be considered in the SPH discretization of Eqs. (1) and (2). In the framework of astrophysical simulation where the density of the fluid can vary enormously, there are different formulations in the literature which take into account particles with different smoothing lengths [14,27]. For incompressible flow Bonet and Rodríguez-Paz [2] derived the SPH discretization of the Navier–Stokes equations starting from the Euler–Lagrange equation of motion and using a conventional SPH density summation (introduced later in Section 3). This formulation has the advantage of ensuring the conservation of momentum, but it is not computationally efficient because the particle densities need to be updated before the particle accelerations can be computed. This means that, to update particle accelerations, two iterations to calculate all particle interactions are needed. In addition, it is well known that calculating the density by a simple summation for the Navier–Stokes equations leads to a non-physical drop of the density near a free surface due to kernel truncation effect. To overcome these issues a novel formulation for the pressure force term in the momentum equation, which conserves the momentum and is more computationally efficient, is presented herein. The key idea is to update the density by using the SPH discretization of the mass continuity equation (1) in place of the SPH density summation:

$$\rho_i = \sum_j m_j W_i(\mathbf{x}_j, h_j) \tag{4}$$

where $W_i(\mathbf{x}_j, h_j)$ is the kernel function, m_j and h_j are the mass and the smoothing length of the j-th particle.

The SPH discretization of the pressure gradient term is derived consistently starting from the Euler–Lagrange equation. This ensures that:

- the momentum is conserved also in the presence of variable smoothing length, h.
- the efficiency of the classical SPH formulation such as in [13] is retained.

The derivation of the formulation is reported in Appendix A, and the final SPH discretization of Eq. (1) is:

$$\frac{d\rho}{dt} = -\rho_i\sum_j \frac{m_j}{\rho_j}(\mathbf{v}_j - \mathbf{v}_i)\cdot\nabla W_j(x_i, h_j) \tag{5}$$

Note that the smoothing length used in the previous equation is h_j, and not the smoothing length of the i-th particle, h_i. This has been shown to be more accurate in the presence of particle splitting [35]. The pressure gradient term of Eq. (2) is discretized as follows:

$$\frac{1}{\rho}\nabla p = \sum_j \frac{m_j}{\rho_j\rho_i}[p_i\nabla W_j(x_i, h_j) - p_j\nabla W_i(x_j, h_i)] \tag{6}$$

Finally, the laminar diffusion term is discretized as reported in [22]:

$$v_0\nabla^2\mathbf{v} = \sum_j m_j\left[\frac{4v_0 r_{ij}\nabla\overline{W}_{ij}}{\rho_{ij}r_{ij}^2}\right]\mathbf{v}_{ij} \tag{7}$$

where v_0 is the kinematic viscosity of the fluid, $r_{ij} = |\mathbf{x}_i - \mathbf{x}_j|$, $\mathbf{v}_{ij} = \mathbf{v}_i - \mathbf{v}_j$, $\rho_{ij} = 0.5(\rho_i + \rho_j)$, and $\nabla\overline{W}_{ij} = 0.5[\nabla W_j(\mathbf{x}_i, h_j) + \nabla W_i(\mathbf{x}_j, h_i)]$. It is well known that in SPH methods a diffusive term needs to be added in order to stabilize the numerical scheme [13]. The most widely used formulation is the artificial viscosity in the momentum equation proposed by Monaghan [25]. To stabilize the SPH numerical scheme Molteni and Colagrossi [24] proposed an alternative approach which consists of adding a diffusivity term in the continuity equation; this formulation is adapted in the present work for particles with different mass and smoothing length. To include this term the continuity equation (5) is therefore modified as follows:

$$\frac{d\rho}{dt} = -\rho_i\sum_j \frac{m_j}{\rho_j}(\mathbf{v}_j - \mathbf{v}_i)\cdot\nabla W_j(x_i, h_j) + \xi h_i c_0\sum_j \frac{m_j}{\rho_j}\psi_{ij}\nabla W_j(x_i, h_j) \tag{8}$$

where ξ is the artificial density diffusion parameter which has to be tuned according to the flow characteristics, c_0 is the initial speed of sound and the term ψ_{ij} is defined as:

$$\psi_{ij} = \left(\frac{\rho_j}{\rho_i} - 1\right)\frac{\mathbf{x}_i - \mathbf{x}_j}{r_{ij}^2 + 0.01h_i^2} \tag{9}$$

Recently Xu et al. [38] and Shadloo et al. [32] showed that introducing a particle shifting correction improves remarkably the accuracy of both incompressible and weakly compressible SPH schemes and prevents errors due to irregular particle distributions. Lind et al. [20] further improved the shifting formulation for free-surface flows by suggesting a shift correction based on the Fick's law. The key idea of the particle shifting is to modify the particle positions, $\mathbf{x}$, as follows:

$$\frac{d\mathbf{x}_i}{dt} = \mathbf{v}_i + \boldsymbol{\delta}_i \tag{10}$$

where $\boldsymbol{\delta}_i$ is the i-th particle shifting vector. We remark that in the shifting algorithm used in the present work only the particle position $\mathbf{x}_i$ is modified whereas the particle velocity $\mathbf{v}_i$ is not modified, therefore exact momentum conservation is maintained. Alternatively it may be argued that velocity should be interpolated to represent more accurately the original velocity field [38], while sacrificing exact momentum conservation. This was in fact tested but had negligible effect on the results.

In the formulations presented in [38,32] the shifting vector $\boldsymbol{\delta}_i$ is modified according to the anisotropy of the particle distribution. This is effective if all the particles have the same size, but it has to be modified if particles with variable mass are used. Herein, the following formulation for the particle shifting vector is proposed: