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A finite element method for a three-field formulation of linear elasticity based on biorthogonal systems

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ABSTRACT

We consider a mixed finite element method based on simplicial triangulations for a three-field formulation of linear elasticity. The three-field formulation is based on three unknowns: displacement, stress and strain. In order to obtain an efficient discretization scheme, we use a pair of finite element bases forming a biorthogonal system for the strain and stress. The biorthogonality relation allows us to statically condense out the strain and stress from the saddle-point system leading to a symmetric and positive-definite system. The strain and stress can be recovered in a post-processing step simply by inverting a diagonal matrix. Moreover, we show a uniform convergence of the finite element approximation in the incompressible limit. Numerical experiments are presented to support the theoretical results.

1. Introduction

It is well known that low-order finite elements with quadrilaterals, hexahedra or simplices constructed from a standard displacement-based formulation of a nearly incompressible elasticity problem exhibit poor performance, in the form of poor coarse-mesh approximations and the locking effect, in which they do not converge uniformly with respect to the Lamé parameter λ. Some relevant works in a substantial literature on the subject include [1,9,11,39].

An important approach for eliminating the locking effect is to use a mixed method. Mixed formulations are generally obtained by formulating a saddle-point problem with additional unknown variables. The linear elasticity problem can be formulated in mixed form in many different ways, see [15,10]. The resulting formulation not only provides an approach to alleviating the locking effect, but may also be used to compute accurately other variables of interest – sometimes called dual variables. These include the stress or pressure in elasticity, while for the Poisson equation the gradient of the solution may be of interest. In standard formulations these additional variables have to be obtained a posteriori by differentiation, potentially resulting in a loss of accuracy.

One of the most popular mixed formulations in elasticity is the Hellinger–Reissner formulation, which is based on the stress and displacement as unknown variables. A stable discretization of the stress-displacement formulation of the elasticity problem requires the construction of a compatible pair of finite element spaces: one for the space of stresses, which are symmetric tensor fields, and the other for the displacements, which are vector fields. The pair of finite element spaces should also be compatible in the sense that they satisfy a suitable inf–sup condition [15,11]. Such a compatible stable pair of finite element spaces using polynomial shape functions was first presented in [4] for triangles and in [2] for rectangles in the case of plane elasticity. It is interesting to note that 24 degrees of freedom are needed for the triangular case and 45 for the rectangular case. These elements are therefore expensive in the two-dimensional case, and their use in three-dimensional elasticity would be prohibitive.

Another popular mixed formulation of elasticity used to overcome the locking effect is the three-field formulation commonly known as the Hu–Washizu formulation [21,43], which was first introduced by Fraeijs de Veubeke [18]. In this formulation, the unknown variables are displacement, stress and strain. The formulation incorporates weak statements of the equation of equilibrium, the strain-displacement equation, and the elasticity relation. The Hu–Washizu formulation has been used frequently to obtain locking-free methods in linear and non-linear elasticity based on bilinear or trilinear finite elements on quadrilaterals or

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hexahedra [40,39,20,22,23,37]. The mathematical analysis for well-posedness of the Hu–Washizu formulation for nearly-incompressible elasticity has been investigated in [31], where it has been shown that a modified version of of the Hu–Washizu formulation is more amenable to obtaining uniform convergence of the finite element approximation in the nearly incompressible regime. However, the analysis is restricted to a class of quadrilateral meshes. Many existing methods such as the assumed strain method, assumed stress method, mixed enhanced method, strain gap method, B-bar method, etc. have been shown in [16] to be special cases of the modified Hu–Washizu formulation.

The objective of this work is to present an extension of the finite element analysis of the three-field formulation to simplicial triangulations. We start with the stabilized Hu–Washizu formulation presented in [29], where the formulation is used to analyze the stability and convergence of the average nodal strain formulation. In this work we introduce a new discretization based on the use of a pair of finite element bases for the stress and strain that form a biorthogonal system. That is, each component of the strain is discretized by the standard linear finite element space, whereas the discrete space of stresses is spanned by basis functions which form a biorthogonal system with the standard finite element space. The biorthogonality relation is an important component of the formulation, inasmuch as it allows the strain and stress to be statically condensed out of the system. The static condensation leads to a reduced system, which is symmetric and positive-definite. The uniform convergence of the finite element solution is shown by using an analysis similar to that in [29]. However, in contrast to [29], uniform convergence is shown without assuming the full H^2-regularity of the solution. Moreover, we prove a priori error estimate for the stress and present a set of numerical results that illustrate the performance and properties of the proposed formulation.

A finite element method using a biorthogonal system for a displacement–pressure formulation of linear elasticity has been presented in [27,28,32]. Although the finite element approximation converges uniformly in the nearly incompressible case, stress cannot be directly computed in these formulations.

There are a few publications devoted to the analysis of enhanced strain techniques for simplicial elements. For example, the mixed enhanced formulation is extended to simplicial meshes in [42]. However, the formulation in [42] is derived using the mini element, requiring that the pressure variable be continuous. Similar enhanced strain methods are discussed in [34,5]. Recovering a displacement-based formulation is not so straightforward.

The structure of the rest of the paper is as follows. In the next section, we fix some notation and briefly recall the standard and the mixed formulations of linear elasticity. We introduce our finite element discretization in Section 3. Section 4 is devoted to the mathematical analysis of the discrete problem. In this section, we show that the finite element approximation converges optimally to the true solution without assuming the full H^2-regularity of the solution, and that convergence does not depend on the Lamé parameter λ. This proves that the method does not exhibit locking in the nearly incompressible regime. Section 5 is devoted to a number of numerical examples which illustrate the performance of the method. Finally, some conclusions are presented in Section 6.

2. Governing equations and weak formulation

Vector- and tensor- or matrix-valued functions will be written in boldface form. The scalar product of two tensors or matrices $\boldsymbol{d}$ and $\boldsymbol{e}$ will be denoted by $\boldsymbol{d} : \boldsymbol{e}$, and is given by $\boldsymbol{a} : \boldsymbol{b} = a_{ij}b_{ij}$, the summation convention on repeated indices being invoked.

We start with the boundary value problem of homogeneous and isotropic linear elastic body occupying a bounded domain Ω in $\mathbb{R}^d, d \in \{2,3\}$ with Lipschitz boundary Γ. Let $L^2(\Omega)$ be the set of all square-integrable functions in Ω, and $\boldsymbol{S} := \{\boldsymbol{d} \in L^2(\Omega)^{d\times d} : \boldsymbol{d}$ is symmetric$\}$ is the set of symmetric tensors in Ω with each component being square-integrable. For a prescribed body force $\boldsymbol{f} \in L^2(\Omega)^d$, the governing equilibrium equation in Ω is

$$-\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f} \tag{1}$$

with $\boldsymbol{\sigma}$ being the symmetric Cauchy stress tensor.

The strain $\boldsymbol{d}$ is related to the displacement through the relation

$$\boldsymbol{d} = \boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla\boldsymbol{u} + [\nabla\boldsymbol{u}]^t), \tag{2}$$

in which $\boldsymbol{\varepsilon}$ is the infinitesimal strain.

Assuming isotropic linear elastic behavior the constitutive relation is given by

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{d} := \lambda(\operatorname{tr}\boldsymbol{d})\mathbf{1} + 2\mu\boldsymbol{d}, \tag{3}$$

where $\mathcal{C}$ denotes the fourth-order elasticity tensor, $\mathbf{1}$ is the identity tensor, and λ and μ are the Lamé parameters. We assume that the body occupying the domain Ω is homogeneous, and λ and μ are positive constants[1]. We focus on the problem of uniform approximation of finite element approximations in the incompressible limit, which corresponds to $\lambda \to \infty$. The inverse of (3) is given by

$$\boldsymbol{d} = \mathcal{C}^{-1}\boldsymbol{\sigma} = \frac{1}{2\mu}\left(\boldsymbol{\sigma} - \frac{\lambda}{2\mu + d\lambda}(\operatorname{tr}\boldsymbol{\sigma})\mathbf{1}\right).$$

We assume that the displacement satisfies the homogeneous Dirichlet boundary condition

$$\boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma. \tag{4}$$

Introducing the Sobolev space $\boldsymbol{V} := [H_0^1(\Omega)]^d$ of displacements with standard inner product $(\cdot,\cdot)_{1,\Omega}$, semi-norm $|\cdot|_{1,\Omega}$, and norm $\|\cdot\|_{1,\Omega}$, see, e.g., [13], we define the bilinear form $A(\cdot,\cdot)$ and the linear functional $\ell(\cdot)$ by

$$A : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}, \quad A(\boldsymbol{u},\boldsymbol{v}) := \int_\Omega \mathcal{C}\boldsymbol{\varepsilon}(\boldsymbol{u}) : \boldsymbol{\varepsilon}(\boldsymbol{v})dx,$$

$$\ell : \boldsymbol{V} \to \mathbb{R}, \quad \ell(\boldsymbol{v}) := \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,dx.$$

Then the standard weak form of the linear elasticity problem is as follows: given $\ell \in \boldsymbol{V}'$, find $\boldsymbol{u} \in \boldsymbol{V}$ that satisfies

$$A(\boldsymbol{u},\boldsymbol{v}) = \ell(\boldsymbol{v}), \quad \boldsymbol{v} \in \boldsymbol{V}. \tag{5}$$

Here $A(\cdot,\cdot)$ is symmetric, continuous, and $\boldsymbol{V}$-elliptic due to Korn's inequality. Hence standard arguments can be used to show that (5) has a unique solution $\boldsymbol{u} \in \boldsymbol{V}$. Furthermore, if the domain Ω is convex polygonal or polyhedral we have $\boldsymbol{u} \in [H^2(\Omega)]^d \cap \boldsymbol{V}$, and there exists a constant C independent of λ such that

$$\|\boldsymbol{u}\|_{2,\Omega} + \lambda\|\operatorname{div}\boldsymbol{u}\|_{1,\Omega} \leqslant C\|\ell\|_{0,\Omega}. \tag{6}$$

We refer to [14] for a proof of the a priori estimate (6) for two-dimensional linear elasticity and [25] for three-dimensional linear elasticity. In order to derive a suitable mixed formulation for the strain–displacement formulation of linear elasticity, we start with the following minimization problem. The variational formulation of the linear elastic problem with homogeneous Dirichlet boundary condition can be written as the following problem:

$$\min_{\substack{(\boldsymbol{u},\boldsymbol{d}) \in \boldsymbol{V}\times\boldsymbol{S} \\ \boldsymbol{d} = \boldsymbol{\epsilon}(\boldsymbol{u})}} \frac{1}{2}\int_\Omega \boldsymbol{d} : \mathcal{C}\boldsymbol{d}dx - \ell(\boldsymbol{u}). \tag{7}$$

[1] This assumption is somewhat stronger than required for the standard elasticity problem where one assumes $\mathcal{C}$ to be pointwise stable and hence $\mu > 0$ and $\lambda > -2/3\mu$. Our interest here, however, is the problem of quasi-incompressible ealasticity where $\lambda \to \infty$.