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Critical softening in Cam-Clay plasticity: Adaptive viscous regularization, dilated time and numerical integration across stress–strain jump discontinuities

R. Conti [a,*], C. Tamagnini [b], A. DeSimone [a]

[a] SISSA-International School for Advanced Studies, Via Bonomea 265, 34136 Trieste, Italy
[b] Università degli Studi di Perugia, Dipartimento di Ingegneria Civile e Ambientale, Via G. Duranti 93, Perugia, Italy

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ABSTRACT

Within the framework of continuum mechanics, the mechanical behaviour of geomaterials is often described through rate-independent elastoplasticity. In this field, the Cam-Clay models are considered as the paradigmatic example of hardening plasticity models exhibiting pressure dependence and dilation-related hardening/softening. Depending on the amount of softening exhibited by the material, the equations governing the elastoplastic evolution problem may become ill-posed, leading to either no solutions or two solution branches (critical and sub-critical softening). Recently, a method was proposed to handle subcritical softening in Cam-Clay plasticity through an adaptive viscoplastic regularization for the equations of the rate-independent evolution problem. In this work, an algorithm for the numerical integration of the Cam-Clay model with adaptive viscoplastic regularization is presented, allowing the numerical treatment of stress–strain jumps in the constitutive response of the material. The algorithm belongs to the class of implicit return mapping schemes, slightly rearranged to take into account the rate-dependent nature of inelastic deformations. Applications of the algorithm to standard axisymmetric compression tests are discussed.

1. Introduction

Many granular materials, such as rocks, concrete, dense sands and stiff clays, exhibit strain softening when subjected to intense shear deformation. This behaviour, which usually manifests itself as a macroscopic loss of strength after a peak load level has been reached, may lead to both spatial (strain localization) and time (critical softening) discontinuities in the response of the material. The first phenomenon occurs when deformations in solids localize into narrow bands [40], while critical softening is related to the loss of test controllability and the onset of snap-back processes during displacement controlled tests [43,17]. In this condition, jumps in the stress–strain behaviour of the material occur and the response of the material evolves faster than the applied loading conditions.

Within the framework of continuum mechanics, the mechanical behaviour of geomaterials is often described through rate-independent elastoplasticity. In this field, the Cam-Clay family of models [32,34,31] are considered as the paradigmatic example of hardening plasticity models exhibiting pressure dependence and dilation-related hardening/softening. As far as rate-independent elastoplasticity is concerned, both localization and critical softening phenomena have been recognized to emerge as local instabilities in the constitutive equations, leading to the ill-posedness of the evolution problem and a non-uniqueness in the incremental response of the material.

Strain localization has been widely analyzed in the literature both from a mathematical [30,3] and a numerical [21,24] point of view. The onset of localization is characterized by the loss of hyperbolicity of the dynamical equations of motion, resulting in the fact that the wave speeds vanish or become imaginary. As a consequence, numerical solutions of localization problems carried out by adopting rate-independent plasticity models suffer from pathological mesh dependence and length-scales effects. Many strategies have been proposed in the literature to model strain-softening behaviour even beyond the onset of localization. Among these regularization techniques, viscoplasticity has been recognized to provide a valuable framework for the analysis of strain localization in solids [21,24,42,29]. One main obstacle in applying viscoplastic regularization techniques to materials exhibiting strong softening response is the possible occurrence of time discontinuities, which emerge in critical softening conditions.

As pointed out by many authors [23,19,7], the inception of critical softening in a strain controlled process corresponds to the vanishing of the determinant of the elastoplastic compliance

* Corresponding author.
E-mail address: rconti@sissa.it (R. Conti).

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matrix or, equivalently, to the vanishing of the so-called *plastic modulus*, K_p. In this condition, for a given strain rate $\dot{\boldsymbol{\varepsilon}}$, the corresponding stress rate $\dot{\boldsymbol{\sigma}}$ goes to infinity and uniqueness of the solution for the rate-independent elastoplastic evolution problem is lost. It is worth noting that, in the literature, the plastic modulus is always assumed to be strictly positive *a priori*. This assumption, which ensures the positiveness of the consistency parameter during a plastic process and the well-posedness of the rate-independent evolution problem, clearly poses a restriction on the allowable amount of material softening that the constitutive model can handle.

Only recently, the possibility to guarantee the well-posedness of the evolution problem even beyond the onset of critical conditions has been investigated, with reference to Cam-Clay plasticity [9,12,11]. It was demonstrated that, by adopting a Duvaut–Lions viscoplastic regularization [15] for the equations of the rate-independent evolution problem, a unique solution exists in the whole space of the admissible stress states [10]. The heuristic idea is that, in the limit as the viscous regularization parameter, τ, goes to zero, two different regimes may occur during plastic loading, depending on the sign of the plastic modulus. For all initial conditions corresponding to which the plastic modulus remains positive, and then the inviscid equations are well-posed, the viscous solution is continuous and tends to the solution of the rate-independent problem as $\tau \to 0$. In this regime, which we will refer to as *slow dynamics*, the response of the material takes place at the same time scale at which the loading conditions evolve. On the other hand, if the plastic modulus vanishes or becomes negative, the limit solution for the viscoplastic problem is discontinuous. To study the evolution of both the stress state and the internal variable along the jump, i.e. to follow the instantaneous response of the material to the applied deformation, a *dilated time* $s := \frac{1}{\tau}t$ may be used to rescale the viscoplastic evolution equations and to compute the state of the material at the end of the jump. We will refer to this rescaled evolution problem as the *fast dynamics* regime.

In this work we focus on the numerical treatment of time instabilities related to critical and subcritical softening in Cam-Clay plasticity models. More specifically, we present an algorithm for the numerical integration of the evolution equations of the regularized viscoplastic Modified Cam-Clay model [31]. The proposed algorithm belongs to the class of implicit return mapping schemes, slightly rearranged to take into account the viscous nature of plastic deformations [37]. Two different strategies have been used to integrate the viscous equations in the slow dynamics regime and their rescaled version along the jumps. As far as the slow dynamics is concerned, the well-conditioned structure of the adopted algorithm allows to recover exactly the return-mapping scheme for the rate-independent problem, in the limit as $\tau \to 0$. On the other hand, by exploiting the mathematical properties of the governing equations, the same structure is essentially preserved also for the integration in the fast dynamics regime. In this case, however, time discontinuities (of stress, plastic deformations and internal variable) are generated in the limit as $\tau \to 0$. This discussion highlights one of the main differences of our approach with respect to viscoplastic regularization schemes already discussed in the literature [21,42]. While in these cases the (regularizing) viscous perturbation is always present, in our case it is invoked *only* when critical softening conditions occur, and it is switched off otherwise. For this reason, our regularization strategy can be described as *adaptive*.

A comprehensive description of return mapping strategies for the integration of both rate-independent and rate-dependent elastoplastic evolution problems can be found in the literature (see e.g. [36,13]). A key ingredient in return mapping schemes is the use of an operator split strategy for the original evolution problem, leading to the so-called closest-point projection approximation [36]. Briefly, the final stress and internal variables are computed from an initial trial elastic state by solving the plastic evolution equations whenever the plastic constraint is activated. By adopting an implicit approximation of the governing equations, the evolution problem is then reduced to a system of non-linear algebraic equations, which is solved customarily with a Newton–Raphson iterative procedure.

As far as the rate-independent problem is concerned, the unknowns to be computed during the integration are the stress (or elastic strains), the internal variables and the discrete consistency parameter, which can no longer be assumed positive *a priori*. Just as for the continuum case, well-posedness of standard numerical algorithms is guaranteed in this case only as long as the discrete plastic multiplier is positive.

In the rate-dependent problem, the stress state can lie outside the yield surface during a plastic process and the consistency condition is no longer necessary. However, by exploiting the variational structure of the rate-dependent Cam-Clay constitutive equations for the slow dynamics regime, it is possible to show that an analogous Lagrange multiplier emerges naturally from the corresponding return mapping scheme, see Section 5. Moreover, the well-conditioned structure of the algorithm allows us to show that this Lagrange multiplier tends to the rate-independent consistency parameter in the limit as $\tau \to 0$.

As for the continuum case, jumps in the viscous solution can be detected in the numerical algorithm by the sign of the Lagrange multiplier: as long as the discrete Lagrange multiplier is positive, the return mapping scheme for the slow dynamics remains well-posed and the viscous solution evolves without discontinuities; on the other hand, if the solution converges to a negative value of the Lagrange multiplier during a time step, then a jump occurs in the stress–strain response of the material and the equations of the fast dynamics have to be integrated to follow the evolution of the stress state along the discontinuity; at the end of the jump, the discrete Lagrange multiplier becomes again positive and the solution returns to evolve in slow dynamics conditions.

The paper is organized as follows. After the introduction of the main notation conventions (Section 2), in Section 3 we recall the existence and uniqueness conditions for the solution of the rate-independent elastoplastic evolution problem. In Section 4 the Modified Cam Clay model is briefly reviewed and properties of the solution for its viscoplastic regularization are outlined. The return mapping algorithm we propose for the numerical integration of the regularized viscoplastic model is presented in Section 5. Results from numerical tests are given in Section 6, where the ability of the algorithm to handle instabilities due to strain softening is assessed by means of a series of single-element tests.

2. Notation

The usual sign convention of soil mechanics (compression positive) is adopted throughout. Following standard notation, bold-face letters denote vectors and second-order tensors, while blackboard-bold symbols denote fourth-order tensors. Accordingly, $\mathbf{I}$ and $\mathbb{I}$ are the second-order and the fourth-order identity tensor respectively. For any two vectors $\mathbf{v}, \mathbf{w}$ the scalar product is defined as: $\mathbf{v} \cdot \mathbf{w} := v_i w_i$, and the dyadic product as: $[\mathbf{v} \otimes \mathbf{w}]_{ij} := v_i w_j$. Accordingly, for any two second-order tensors $\mathbf{X}, \mathbf{Y}$, $\mathbf{X} \cdot \mathbf{Y} := X_{ij} Y_{ij}$ and $[\mathbf{X} \otimes \mathbf{Y}]_{ijkl} := X_{ij} Y_{kl}$. The Euclidean norm of a second order tensor $\mathbf{X}$ is defined as $\|\mathbf{X}\| := \sqrt{\mathbf{X} \cdot \mathbf{X}}$. For any vector field $\mathbf{v}$, $\nabla \mathbf{v}$ denotes the spatial gradient of $\mathbf{v}$.

In the representation of stress and strain states, the following invariant quantities will be used in the paper:

$$p := \frac{1}{3}\mathrm{tr}(\boldsymbol{\sigma}); \quad q := \sqrt{\frac{3}{2}}\|\mathbf{s}\|; \quad S := \sin(3\theta) := \sqrt{6}\,\frac{\mathrm{tr}(\mathbf{s}^3)}{[\mathrm{tr}(\mathbf{s}^2)]^{3/2}} \tag{1}$$