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A multiplicative approach for nonlinear electro-elasticity

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ABSTRACT

The recent interest in dielectric elastomers has given rise to a pressing need for predictive non-linear electromechanical coupling models. Since elastomers behave elastically and can sustain large deformations, the constitutive laws are naturally based on the formulation of adequate free energy functions. Due to the coupling, such functions include terms which combine the strain tensor and the electric field. In contrast to existing frameworks, this paper proposes to establish the electromechanical coupling by the multiplicative split of the deformation gradient into a part related to the elastic behavior of the material and further one which describes the deformation induced by the electric field. Already available and well tested functions of elastic free energy functions can be immediately deployed without any modifications provided the argument of the function is the strain tensor alone which in turn is defined by the elastic part of the deformation gradient only. An appropriate constitutive relation is formulated for the electrically induced part of the deformation gradient. The paper discusses in depth such a formulation. The approach is elegant, straightforward and above all, provides clear physical insight. The paper presents also a numerical formulation of the theoretical framework based on a meshfree method. Various numerical examples of highly non-linear coupled deformations demonstrate the potential and strength of the theory.

1. Introduction

In recent years, functional or active materials have played an increasingly important role for the design of advanced and smart structures as well as intelligent and micro-electromechanical systems (MEMS). This has propelled interdisciplinary studies on the modeling and application of related constituents; for an overview the reader is referred to the monograph [38]. Transducing materials are of special interest, since they serve as elementary (linear) units such as sensors and actuators well-suited for example for control components monitoring damage in solid structures via adhesively bonded piezoelectric patches [5]. Amongst these kinds of smart materials are smart hydrogels, ferroelectric polymers, piezoelectric polymers, electrostatic polymers, ionic polymer–metal composites and conducting polymers which are collectively known as dielectric elastomers or electro-active polymers (EAP). EAP have been discovered to be very useful, because, in contrast to piezoelectric ceramics, they are less critical with regard to deformability and formability. The material can be simply cast into shapes which meet the design and modality requirements and are resilient as well as light-weight. EAP are highly compliant and typically require low actuation voltage (in the order of several volts) to show high strain output on the order of hundred of percent, but are nearly incompressible. Consequently, the material behavior is characterized by high non-linearity. The field of application as actuators, sensors and energy harvesting devices shows a broad versatility ranging from bio- and micro-manipulation, biomimetic robotics, prosthetics, and smart structures [27]. Bar-Cohen [3] reported on EAP actuators acting as artificial muscles. Biological muscles are considered to be highly optimized systems, since they are fundamentally the same for all animals, and morphological changes between species are small.

The properties of EAP have been intensively studied and various models have been developed to capture their behavior in specific applications. Li et al. [19] developed a model for an application to electric-sensitive hydrogels which is called the refined multi-effect-coupling electric-stimulus (rMECe) model, wherein the fixed charge density and finite deformation were considered. An overview about ionic polymer–metal composites (IPMCs) was given by Shahinpoor et al. [33]. Therein, an introduction to ionic polymer metal composites as biomimetic sensors and actuators were discussed. The same author [32], presented a simple phenomenological continuum theory for the non-homogeneous deformation of ionic polymeric gels in an electric field characterized by electro-elastic bending. An experimental study on dielectric

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elastomers undertaken by Zhao et al. [46] demonstrated that the homogeneous deformation of a layer of a dielectric elastomer subjected to a voltage could be unstable, giving way to an inhomogeneous deformation, such that two regions coexisted within the layer. Keplinger et al. [15] presented an electrical measurement technique for obtaining information on the transient strain in the actuator and analyzed the behavior of the actuator in safe and failure operation regimes. The method also featured the strain-dependent measurement of the electrode resistance. Similar electromechanical loading experiments were conducted that captured the large deformation dynamic behavior of axisymmetric dielectric elastomer membranes subjected to dynamic electrical loading and dynamic mechanical loading experiments [12]. A model for a biaxially pre-strained circular actuator was proposed to characterize the electromechanical behavior of dielectric elastomers [45]. In this approach, the deformation of the actuator for a given activation voltage depended on the three-dimensional mechanical behavior of the film. This model was embedded into the finite element formulation, wherein time dependent material behavior was then included by using hyperelastic–viscoelastic models with material parameters determined from relaxation and tensile tests.

The general non-linear electro-elasticity equations were developed by Toupin [41] and Maugin [21]. Constitutive relations of non-linear electro-elastic material were formulated by Voltairas et al. [42] as well as [9]. Vu et al. [43] considered a variational approach for realistic modeling of EAP, and provided the finite element implementation details. Suo et al. [40] proposed a principle of virtual work which accounted for the criticism of [22,25] on existing theories based on the definition of electric force. In [7] different electromechanical energy formulations and a comprehensive variational framework were presented and the latter compared with works by Refs. [41,11]. An extension of the theories of electro-active polymers into the realm of generalized continua was recently achieved by Skatulla et al. [35]. Such a continuum theory naturally provides the means to address scale effects. Bustamante et al. [8] discussed the interaction between electric fields and deformable media within the quasi-static context. The governing equations were expressed in terms of different measures of stress and their relation to a corresponding principle of virtual work and variational principle were noted. It was summarized that the solution based on the total energy density function was indeed appropriate for electro-active elastomers. Dorfmann and Ogden [10] centered their research on formulating a nonlinear constitutive framework that described the linearized response of electro-elastic solids superimposed on a state of finite deformation in the presence of an electric field. It showed that stability is crucially dependent on the magnitudes of the electromechanical coupling parameters in the constitutive equation. A recent review by Suo [39] focused on the theory of dielectric elastomers developed within continuum mechanics and thermodynamics context and motivated by molecular scans and empirical observations.

By the very fact that large elastic deformations can be sustained, electro-active polymers lend themselves to frameworks based on the formulation of a free energy function. One of the pressing issues of many such formulations relates to the adequate consideration of coupling terms where the strain tensor – usually considered to be the right *Cauchy–Green* deformation tensor $\mathbf{C}$ – and the electric field $\mathbf{e}$ become intertwined. In contrast to existing formulations this paper embarks on a new formulation – originally introduced by Skatulla et al. [37] and Sansour et al. [30] – which is based on the idea of splitting the deformation gradient $\mathbf{F}$ in a multiplicative fashion into an elastic (mechanically induced) part and a second part induced by the electric field: $\mathbf{F} = \mathbf{F}_{mech}\mathbf{F}_{elec}$. The advantages of such a decomposition become immediately apparent: (i) existing well established and tested formulations of free energy functions for elastic polymers can be directly adopted provided the function is formulated in terms of strain tensors only based on $\mathbf{F}_{mech}$. (ii) The coupling is directly given by an appropriate constitutive law for $\mathbf{F}_{elec}$ in terms of the electric field. Such a formulation is transparent, straightforward and can be extended to arbitrary non-linearities with respect to the electric field. (iii) Physical insight regarding the nature of the constitutive law is immediately provided. We note that such a decomposition has been extensively considered in the theory of plastic deformations; see e.g. [6,18,31,34]. The nature of the constitutive law, however, is completely different, since plasticity is a dissipative process where the inelastic part of the deformation gradient is defined by means of an integration process. The multiplicative decomposition of the deformation gradient was also proposed in the context of ferroelectrics by Rosato and Miehe [26] considering hysteresis at large strains and in thermoelasticity differentiating between elastic and thermal parts, e.g. in [44].

Further, having established the theoretical framework, the paper provides a numerical formulation based on a meshfree method. While the choice of the meshfree method is adequate in terms of approximation smoothness and consistency, an alternative discretization method such as the finite element method could be utilized as well.

The paper is organized as follows: In Section 2 the basics of electrostatics and the multiplicative electromechanical coupling theory are outlined. Based on this theory an electromechanically coupled variational principle is presented in Section 3. Finally, in Section 4 the proposed electromechanical variational formulation is applied to three numerical examples modeling non-linear hyperelastic material.

2. Electromechanics theory

This section is not meant to be a comprehensive introduction in electro-mechanics but rather to present the basic electro-mechanical fields and equations in electrostatics. For further details the reader is referred to [14,16].

2.1. Basics

Let $\mathbb{E}(3)$ be the Euclidian vector space and $\mathcal{B} \subset \mathbb{E}(3)$, where $\mathcal{B}$ is a three-dimensional manifold defining a material body. A motion of $\mathcal{B}$ is represented by a one parameter non-linear deformation mapping $\boldsymbol{\varphi}_t : \mathcal{B} \rightarrow \mathcal{B}_t$, where $t \in \mathbb{R}$ is the time and $\mathcal{B}_t$ is the current configuration at time t. Accordingly, each material point $\mathbf{X} \in \mathcal{B}$ is related to its placement $\mathbf{x}$ in the spatial configuration $\mathcal{B}_t$ by the mapping

$$\mathbf{x} = \boldsymbol{\varphi}(\mathbf{X}, t), \tag{1}$$

In what follows and without loss of generality we identify the body $\mathcal{B}$ with its undeformed reference configuration at a fixed time t_0. The deformation map possesses an invertible linear tangent map $\mathbf{F} = \mathrm{Grad}\,\boldsymbol{\varphi}$ denoted by the deformation gradient, where the Jacobian $J = \det \mathbf{F} > 0$. The operator Grad represents the gradient with respect to the reference configuration

$$\mathrm{Grad} := \frac{\partial}{\partial \mathbf{X}}, \tag{2}$$

and for later usage, the equivalent operation in the current configuration is defined as

$$\mathrm{grad} := \frac{\partial}{\partial \mathbf{x}}. \tag{3}$$

The body $\mathcal{B}$ is parameterized by the Cartesian coordinates $X_i,\ i = 1, 2, 3$. Here, and in what follows, Latin indices take the values $1,\ 2$ or 3. As a deformation measure we make use of the right *Cauchy-Green* deformation tensor $\mathbf{C}$ defined by