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Quadratically consistent one-point (QC1) quadrature for meshfree Galerkin methods

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ABSTRACT

A robust and efficient integration method, named quadratically consistent one-point (QC1) scheme, which evaluates the Galerkin weak form only at the centers of background triangle elements (cells) is proposed for meshfree methods using quadratic basis. The strain at the evaluation points is approximated by corrected (smoothed) nodal derivatives which are determined by a discrete form of the divergence theorem between nodal shape functions and their derivatives in Taylor's expansion. We prove that such smoothed nodal derivatives also meet the differentiation of the approximation consistency (DAC). The same Taylor's expansion is applied to the weak form and the smoothed nodal derivatives are used to compute the stiffness matrix. The proposed QC1 scheme can pass both the linear and the quadratic patch tests exactly in a numerical sense. Several examples are provided to demonstrate its better numerical performance in terms of convergence, accuracy, efficiency and stability over other one-point integration methods in the meshfree literature, especially its superiority over the existing linearly consistent one-point (LC1) quadratures.

1. Introduction

One-point quadrature with hourglass controls [1–5] provides tremendous benefit in 2D/3D finite element analysis of engineering problems, since it uses minimum evaluations of the weak form and at the same time removes the spurious zero-energy modes. In contrast, full quadratures generally require 4 and 8 integration points for 2D quadrilateral and 3D hexahedral elements, respectively. The computational savings of one-point quadrature are remarkable, especially for non-linear problems. In addition, it can be applied to nearly incompressible materials while the full quadratures cannot due to their volumetric locking [4,5]. Actually, the benefit of one-point quadrature is not limited to the above. For example, in the crack analysis by the extended finite element method (XFEM) [6,7], one-point quadrature provides significant convenience in cutting elements since it avoids the dilemma when some quadrature points in one element satisfy the fracture criterion whereas the others do not [8,9]. Due to these reasons, one-point quadrature is widely used in finite elements and is already available in many commercial softwares, e.g. ABAQUS [10] and LS-DYNA [11].

On the other hand, meshfree Galerkin methods [12,13] developed in the past twenty years show considerable advantages against the traditional finite element methods, e.g. in large deformation analysis [14] and in adaptive computations [15]. However, accurate integration of the Galerkin weak form in meshfree methods is more difficult than in finite element methods due to the non-polynomial character of meshfree approximations as well as the misalignment between the nodal supports and the integration cells [16]. To ensure a stable solution, higher order Gauss integration is commonly employed in meshfree Galerkin methods, e.g. 4×4 Gauss quadrature for background quadrilaterals is suggested in [12]. This kind of method is very costly and, more importantly, it fails to pass the patch tests exactly as shown in [17].

Several strategies were developed in the literature to accelerate meshfree computations such as the stress-point methods [18–20] and the support integration methods [21–24]. As in finite element methods mentioned above, it is crucial to develop stable and efficient one-point quadrature for meshfree Galerkin methods and many efforts have been devoted to this field. It is noted that one-point quadrature in meshfree literature is usually formulated as nodal integrations in which the evaluation points and the approximation nodes coincide. In essence, nodal integration is one-point quadrature since in each background integration cell only one point, i.e. the node associated to this cell, is used to evaluate the weak form.

Beissel and Belytschko [25] first proposed a nodal integration method with least-square stabilization, in which the square of the residual of the equilibrium equation is introduced to the weak form as a stabilization mechanism to remove spurious singular modes. However, the magnitude of such stabilization term is dependent on a numerical parameter and its proper choice is

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required. Otherwise, the results will be over diffusive or oscillated. Nagashima [26] developed a nodal integration scheme for element-free Galerkin (EFG) method by using a Taylor's expansion technique which is first introduced by Liu et al. in finite element method [27] and in reproducing kernel particle method (RKPM) [28] as well. In this kind of stabilization, Taylor's expansion of the stiffness matrix is employed such that the high order terms as stabilization are introduced in a rational manner and no numerical parameter is involved. This technique was also used by Liu et al. [29] to stabilize the nodal integration in radial point interpolation method (RPIM) and more high order terms are retained. However, Duan and Belytschko [20] showed that Taylor's expansion based stabilization (TEBS) is not adequate for nodal integration of the EFG method with quadratic approximations and therefore considerable oscillations present in the resulting stress fields. Convergence in displacement is also lost for fine discretizations. In addition, no paper reports its performance in patch tests which are very important for convergence in a practical sense. Bonet and Kulasegaram [30], in the context of corrected smooth particle hydrodynamics (CSPH), proposed a nodal integration method with an integration correction which enables the method to pass linear patch test. However, the method still requires a similar least-square stabilization as in [25] and the integration correction needs to be solved in an iterative manner.

Chen et al., in an important work with very high citations [31], proposed a stabilized conforming nodal integration (SCNI) method and strain smoothing is used for stabilization. They showed that the strain smoothing meets the integration constraint (IC) which is essentially the discrete form of the linear patch test condition. So, SCNI can provide even better accuracy than Gauss integration which does not meet the IC, i.e. fails to pass linear patch test exactly. However, as pointed out by Duan et al. in [17], the smoothed strain evaluated at the center of each background integration cell can only reproduce a constant strain field in the cell. Consequently, this kind of one-point quadrature can only provide accuracies and convergence comparable to linear finite elements even if a quadratic approximation is employed. In addition, mild oscillations are observed in stress fields and such spurious instabilities get worse if non-constant body forces present.

As described above, one-point quadrature in meshfree Galerkin methods always appears as a nodal integration scheme. One of the main reasons is, in such a way, the methods are more "truly-meshless" like since only one set of nodes is used. However, the determination of the nodal weights for integration usually requires a "mesh" for the domain, e.g. the Voronoi diagram used in [31].

In this study, we will develop a robust and efficient one-point quadrature for meshfree Galerkin methods with quadratic approximations. The key feature of the proposed method is that it can reproduce a linear strain field which is consistent to the quadratic approximation for the displacement using only one point in each integration cell. Background triangle cells (elements) are used and only their centers are taken as evaluation points. The stabilization comes from correcting the derivatives of the nodal shape functions at the quadrature points according to the consistency framework for the nodal derivatives proposed in [17], which includes the differentiation of approximation consistency (DAC) and the discrete divergence consistency (DDC). The corrected (smoothed) nodal derivatives are computed by the satisfaction of the quadratic DDC. The idea of this paper is the introduction of the Taylor's expansion as in [26] into the quadratic DDC such that this consistency can be satisfied by using only one point in each integration cell. In contrast, three points per cell are employed in [17] to enforce this condition. We further prove that the smoothed derivatives also meet the quadratic DAC and therefore the proposed method is a quadratically consistent one-point (QC1) integration scheme which is the key contribution of this paper. In contrast, the SCNI [31] only meets the linear DDC and does not satisfy the quadratic one. Therefore, this kind of method in this study is named as linearly consistent one-point (LC1) integration scheme. It is noted that the hourglass controls [1–5] were developed for low-order elements and also only provide a linear accuracy. To our knowledge, the proposed QC1 is the first one-point quadrature for meshfree methods with quadratic exactness, i.e. it can pass quadratic patch test exactly.

The paper is organized as follows. The EFG method is briefly reviewed in Section 2. The consistency framework for the nodal derivatives proposed in [17] is described in Section 3. The proposed one-point quadrature QC1 is presented in Section 4. Its quadratic consistency is proved in Section 5 followed by the numerical examples in Section 6 and conclusions in Section 7.

2. Element-free Galerkin (EFG) method

In EFG method, the nodal shape functions $N_I(\mathbf{x})$ are constructed by the moving least-squares (MLS) approximation which originates in data fitting. Here, we follow the generalization of MLS in [32], which can accelerate the computations of the nodal shape functions and their derivatives. The nodal shape functions $N_I(\mathbf{x})$ can be given by

$$N_I(\mathbf{x}) = \mathbf{p}^{\mathrm{T}}(\mathbf{X}_I) w_I(\mathbf{x}) \boldsymbol{\alpha}(\mathbf{x}) \tag{1}$$

where $w_I(\mathbf{x})$ is a weight function which is positive and compactly supported. In this study, we consider the following weight:

$$w_I(\mathbf{x}) = w(\bar{s}) = \begin{cases} 1.0 - 15\bar{s}^4 + 24\bar{s}^5 - 10\bar{s}^6 & \text{for } \bar{s} \leqslant 1, \\ 0, & \text{for } \bar{s} > 1, \end{cases} \tag{2}$$

where $\bar{s} = s/r$, r is the radius of the support, $s = |\mathbf{x} - \mathbf{X}_I|$ the distance from the sampling point $\mathbf{x}$ to the node $\mathbf{X}_I$. In this study, upper case $\mathbf{X}$ is used to denote the approximation node, whereas lower case $\mathbf{x}$ is used to denote an arbitrary point and in most cases it denotes the evaluation point.

$\mathbf{p}(\mathbf{x})$ in Eq. (1) is a vector of base functions which usually includes a complete basis of the polynomials with a given order. Note that the weight function given in Eq. (2) is C^2 with respect to x and y, which is required by the proposed method QC1 as shown in Section 4. The quartic spline used in [17] is only C^1 since its second-order spatial derivatives with respect to x and y are singular at the node, i.e. at $\bar{s} = 0$, and cannot be used in QC1.

The unknown vector $\boldsymbol{\alpha}(\mathbf{x})$ in Eq. (1) is computed by the reproducibility condition which requires that the MLS approximation reproduces exactly the polynomial basis $\mathbf{p}(\mathbf{x})$, i.e.

$$\mathbf{p}(\mathbf{x}) = \sum_I \mathbf{p}(\mathbf{X}_I) N_I(\mathbf{x}) \tag{3}$$

Substitution of Eq. (1) into the above equation gives

$$\mathbf{A}(\mathbf{x}) \boldsymbol{\alpha}(\mathbf{x}) = \mathbf{p}(\mathbf{x}) \tag{4}$$

where

$$\mathbf{A}(\mathbf{x}) = \sum_I \mathbf{p}(\mathbf{X}_I) \mathbf{p}^{\mathrm{T}}(\mathbf{X}_I) w_I(\mathbf{x}) \tag{5}$$

Thus, the nodal MLS shape functions $N_I(\mathbf{x})$ can be computed by Eq. (1) after solving Eq. (4) for the unknown vector $\boldsymbol{\alpha}(\mathbf{x})$.

The standard derivatives of the shape function $N_{I,i}(\mathbf{x})$ are computed by directly differentiating Eq. (1)

$$N_{I,i}(\mathbf{x}) = \mathbf{p}^{\mathrm{T}}(\mathbf{X}_I)[w_{I,i}(\mathbf{x}) \boldsymbol{\alpha}(\mathbf{x}) + w_I(\mathbf{x}) \boldsymbol{\alpha}_{,i}(\mathbf{x})] \tag{6}$$

where the unknown $\boldsymbol{\alpha}_{,i}(\mathbf{x})$ is obtained by directly differentiating Eq. (4)

$$\mathbf{A}(\mathbf{x}) \boldsymbol{\alpha}_{,i}(\mathbf{x}) = \mathbf{p}_{,i}(\mathbf{x}) - \mathbf{A}_{,i}(\mathbf{x}) \boldsymbol{\alpha}(\mathbf{x}) \tag{7}$$