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Consistent tangent operator for cutting-plane algorithm of elasto-plasticity

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ABSTRACT

The paper presents a derivation of the consistent tangent operator (CTO) for the cutting-plane algorithm (CPA). For a class of plasticity models that are suitable to be integrated using CPA, an explicit recursive expression is analytically derived and is updated in each iteration of the CPA integration procedure to yield the final value of the CTO when the CPA is converged.

1. Introduction

A nonlinear boundary value problem (BVP) in continuum solid mechanics is, nowadays, efficiently solved computationally with the finite element method (FEM), and an appropriate integration scheme is adopted to numerically tackle the given material constitutive behavior. With reference to the latter, a wide class of return-mapping algorithms based on an operator-splitting methodology [1] prevails, particularly, in the computational rate-independent plasticity. First efforts were made in 60s with pioneering work of Wilkins [2] who tackled materials obeying the von Mises yield criterion. Many extensions of the originally proposed algorithm were made through the years to take different physical phenomena into account. Krieg and Key [3] extended the Wilkins work with isotropic and kinematic hardening effects. Non-smooth multisurface (visco)plasticity and large deformation formulations were tackled by Simo et al. [4] and Nagtegaal [5], respectively. The same operator-splitting methodology was also applied to extension of the von Mises model with isotropic and kinematic hardening effects with temperature and creep dependency [6]. Simo and Taylor [7] analyzed the plane stress conditions were algorithmic treatment is reduced to the solution of a single non-linear equation. Non-classical models (such as Cap model) and rate-dependent models were also treated by Simo et al. [8]. A special reference to the integration of elasto-plastic dynamic problems was made by Pinsky et al. [9]. A more general formulation embodying both multiplicative and additive strain decomposition theories with accommodating general yield criteria, arbitrary flow and hardening rules was made by Ortiz and Simo [10]. Ortiz and Popov [11] also extended the framework with well-known trapezoidal and midpoint rules to achieve second-order accuracy. A special concern from the viewpoint of mapping back to the yield surface was examined by Yoon et al. [12] who exposed that if the strain increment is not small enough the solution by returning back to the yield

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surface might be difficult to find even though the solution mathematically exists. Consequently, an enhanced iterative solution scheme based on the work of Ortiz and Simo [13] by utilizing the control of the potential residual was developed.

Basically, the operator-splitting methodology follows the additive split of the constitutive equations into the elastic predictor part and, when required, the subsequent plastic corrector part with the elastically predicted state variables iteratively projected back onto the yield surface. Such methods, usually also referred to as elastic predictor-plastic corrector algorithms, are classically performed according to the backward-Euler approach. Because in a simple geometric interpretation the backward-Euler approach results in a projection of the elastic trial state on the closest point of the yield surface, those methods are also referred to as the closest point projection (CPP) methods.

Despite all of their advantages and their general popularity, the CPP methods are most suitable for only simple plasticity models. Namely, in the case of complex constitutive models, the solution may prove to be exceedingly laborious [13], which is due to the evaluation of the plastic flow direction gradients and second-order yield-function derivatives required in the Newton–Raphson solution procedure. To overcome the described shortcomings, an alternative method that is always used within the framework of the operator-splitting methodology was proposed [13,14], with the plastic correction stage performed based on the forward-Euler approach. The method addressed as the cutting-plane algorithm (CPA) does not require a computation of the above described gradients. Consequently, it is simpler to implement into a FEM code and computationally more efficient.

However, along with the above described benefits, the CPA method suffers, according to the relevant literature [15], from a lack of finding a consistent tangent operator (CTO). This operator is necessary to ensure computational efficiency on the global level, where a solution to the nonlinear BVP is sought within the implicit FEM framework by applying the Newton–Raphson iterative solution scheme. In [15] it is also stated that, although the simplicity of the algorithm leads to a very attractive computational scheme for large scale calculations, it appears that exact linearization of the algorithm cannot be obtained in closed form. The derivation seems to be impossible. In their opinion, global solution strategies that involve quasi-Newton methods are required in such a case. Furthermore, some others [16–19] suggest a more convenient approach—the use of the continuum tangent operator as an alternative.

Although the lack of the CTO seems to present a serious limitation, the CPA application is still present in many very recent publications, where an application of other backward-Euler-based return-mapping methods on the complex constitutive equations would be exceedingly laborious [16,20–29].

In this paper, we prove that an analytical derivation of the CTO for CPA is not only possible but also straightforward. The proposed approach results in the analytically derived recursive formulation of the CTO for a general class of plasticity models, but its extension to other constitutive models, e.g. viscoplasticity, represents no additional effort.

The article content is structured in the following way. First, theoretical preliminaries with special attention to the basic equations of a general class of rate-independent plasticity models are given in the form of a newly introduced stacked vector/matrix notation, which makes the derivation of the CTO more suitable. In Section 3, a brief description of the CPA using the new notation is given. For a more detailed insight into the CPA, however, the reader should be referred to [13,15]. In Section 4, several aspects regarding the notion of the CTO, including that of its role in the global scheme of the BVP solution, are addressed first. The section ends with a general analytical derivation of the CTO. To maintain the main beauty of the CPA, which proceeds from the fact that evaluation of the second-order derivatives of the plastic potential function, associated with the plastic flow direction gradients, is not needed, the semi-analytical approach is also proposed. With such an approach elaborated in Section 5, a consistency of the integration scheme and the tangent moduli is preserved, and a generalization to complex constitutive relations (for example, the YLD2004-18p model that [30,31] proposed) can easily be made (Section 6). The performances of the derived consistent vs. continuum tangent operator are finally monitored in Section 7 by considering two case-study simulations.

2. Theoretical preliminaries

2.1. Constitutive modeling in plasticity

Let us review the governing equations of the incremental rate-independent theory of plasticity, which can optionally be extended to embrace other phenomena (e.g. damage, distortional hardening, etc.) for a proper material characterization.

The admissible stress space in an elastic–plastic problem is formulated through the algebraic constraint $\Phi = \Phi(\sigma_{ij}, \sigma_Y, \kappa_1, \ldots, \kappa_p) \leqslant 0$, where $\Phi = 0$ defines the actual yield surface. The yield surface is a function of the actual yield stress $\sigma_Y = \sigma_Y(\varepsilon^p_{eq})$ and internal state variables $\kappa_r,\ r \in \{1, 2, \ldots, p\}$, which are both subject to change under plastic deformation. The material response associated with a change $d\sigma_{ij}$ of the stress tensor σ_{ij} is fully elastic if $\Phi < 0$, whereas in the case of $\Phi = 0$, the material response is certainly elastic, if $d\Phi < 0$, while $d\Phi = 0$ requires further investigation in order to find out whether the material response is elastic or elastic–plastic. To stay focused on the subject of this article, which is devoted exclusively to consideration of the elastic–plastic response, we assume that all conditions required for the elastic–plastic deformation to take place are fulfilled. Thus, the actual stress state is supposed to satisfy the yield criterion

$$\Phi = \Phi(\sigma_{ij}, \sigma_Y, \kappa_1, \ldots, \kappa_p) = 0. \tag{1}$$