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Stabilization of projection-based reduced order models for linear time-invariant systems via optimization-based eigenvalue reassignment

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A B S T R A C T

A new approach for stabilizing unstable reduced order models (ROMs) for linear time-invariant (LTI) systems through an *a posteriori* post-processing step applied to the algebraic ROM system is developed. The key idea is to modify the unstable eigenvalues of the ROM system by moving these eigenvalues into the stable half of the complex plane. It is demonstrated that this modification to the ROM system eigenvalues can be accomplished using full state feedback (a.k.a. pole placement) algorithms from control theory. This approach ensures that the modified ROM is stable provided the system's unstable poles are controllable and observable; however, the accuracy of the stabilized ROM is not guaranteed. To remedy this difficulty and guarantee an accurate stabilized ROM, a constrained nonlinear least-squares optimization problem for the stabilized ROM eigenvalues in which the error in the ROM output is minimized is formulated. This optimization problem is small and therefore computationally inexpensive to solve. Performance of the proposed algorithms is evaluated on two test cases for which ROMs constructed via the proper orthogonal decomposition (POD)/Galerkin method suffer from instabilities.

1. Introduction

As computing power has increased, so has the complexity of multi-physics models. Simultaneously, there has been a continuing push to incorporate uncertainty quantification (UQ) into high-fidelity simulations. Unfortunately, integrating UQ techniques into high-fidelity simulation codes can present an intractable computational burden due to the high-dimensional systems that arise, as well as the need to run these simulations many times in order to explore a space of design parameters or uncertain inputs.

Reduced order modeling is a promising tool that can enable not only UQ, but also on-the-spot decision-making, optimization and/or control. A reduced order model (ROM) is a surrogate model constructed from a full order (high-fidelity) model (FOM) that retains the essential physics and dynamics of the FOM, but has a much lower computational cost. Numerous approaches to construct ROMs exist, from simply running a numerical simulation on a coarser mesh, to surrogates obtained from data-fitting (e.g., Kriging interpolation). More commonly, however, the term "reduced order model" refers to a

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projection-based reduced order model, the subject of the present work. The basic idea of projection-based reduced order modeling is to project the state of a large dimensional space onto a small dimensional subspace that contains the essential dynamics of the system. Examples of projection-based model reduction approaches include proper orthogonal decomposition (POD) [13,14,9], balanced proper orthogonal decomposition (BPOD) [19,11], balanced truncation [16,5], the reduced basis method [15,32], and Krylov-based techniques [31].

In order for a ROM to serve as a viable mathematical model of a physical system of interest, it is important that it preserves certain crucial properties of the original system. Particularly important is that the ROM maintains numerical stability of its underlying physical system, as stability is a prerequisite for the ROM's accuracy and convergence. Some projection-based model reduction techniques give rise to ROMs with an *a priori* stability guarantee. One example of such a method is balanced truncation [16,5]. Unfortunately, the computational cost of this method, which requires the computation and simultaneous diagonalization of infinite controllability and observability Gramians, makes balanced truncation computationally intractable for systems of very large dimensions (i.e., systems with more than 10,000 degrees of freedom (dofs) [12]). Among the most popular model reduction techniques that are computationally tractable for very large systems are the POD method [13,14,9] and the BPOD method [19,11]. In general, these methods lack an *a priori* stability guarantee. In [18], Amsallem et al. suggest that POD and BPOD ROMs constructed for linear time-invariant (LTI) systems in descriptor form tend to possess better numerical stability properties than POD ROMs constructed for LTI systems in non-descriptor form. Although heuristics such as these exist, it is in general unknown *a priori* if a ROM constructed using POD or BPOD will preserve the stability properties of the high-fidelity system from which the model was constructed. Hence, a ROM might be stable for a given number of modes, but unstable for other choices of basis size; see [10,3,4] for examples of this for POD models of compressible flow.

A literature search reveals that approaches for developing stability-preserving projection-based ROMs based on POD and BPOD fall into roughly three categories, overviewed briefly below.

The first category of methods derives (*a priori*) a stability-preserving model reduction framework, often specific to a particular equation set. In [12], Rowley et al. show that Galerkin projection preserves the stability of an equilibrium point at the origin if the ROM is constructed in an "energy-based" inner product. In [3,4], Barone et al. demonstrate that a symmetry transformation leads to a stable formulation for a Galerkin ROM for the linearized compressible Euler equations [3,4] and non-linear compressible Navier–Stokes equations [17] with solid wall and far-field boundary conditions. In [1], Serre et al. propose applying the stabilizing projection developed by Barone et al. in [3,4] to a skew-symmetric system constructed by augmenting a given linear system with its adjoint system. This approach yields a ROM that is stable at finite time even if the solution energy of the full-order model is growing. In [35,40], Sirisup et al. develop a method for correcting long-term unstable behavior for POD models using a spectral viscosity (SV) diffusion convolution operator. The advantage of approaches such as these is they are physics-based, and guarantee *a priori* a stable ROM; the downside is that they can be difficult to implement, as access to the high-fidelity code and/or the governing partial differential equations (PDEs) is often required.

A second category of methods is aimed to remedy the so-called "mode truncation instability". These methods [36–38,23,41], motivated by the observation that higher order modes can give rise to nonphysical instabilities in the ROM system, are often physics-based and minimally intrusive to the ROM. In [23], a ROM stabilization methodology that achieves improved accuracy and stability through the use of a new set of basis functions representing the small, energy-dissipation scales of turbulent flows is derived by Balajewicz et al. The stabilization of ROMs using shift modes and residual modes was proposed in [37,38] by Noack et al. and Bergmann et al. respectively. Other authors, e.g., Terragni et al. [41], have demonstrated that the stability and performance of a ROM can be improved by adapting the POD manifold to the local dynamics.

The third category of approaches are those which stabilize an unstable ROM through a post-processing (*a posteriori*) stabilization step applied to an unstable algebraic ROM system. Ideally, the stabilization only minimally alters the ROM physics, so that the ROM's accuracy is not sacrificed. In [2], Amsallem et al. propose a method for stabilizing projection-based linear ROMs through the solution of a small-scale convex optimization problem. In [22], a set of linear constraints for the left-projection matrix, given the right-projection matrix, are derived by Bond et al. to yield a projection framework that is guaranteed to generate a stable ROM. In [20], Zhu et al. derive some large eddy simulation (LES) closure models for POD ROMs for the incompressible Navier–Stokes equations, and demonstrate numerically that the inclusion of these LES terms yields a ROM with increased numerical stability. In [39], Couplet et al. propose methods for correcting the behavior of a low-order POD-Galerkin system through a coefficient calibration/minimization. A nice feature of these and similar approaches is that they are easy to implement: often the stabilization step can be applied in a "black-box" fashion to an algebraic ROM system that has already been constructed. However, the approaches can give rise to inconsistencies between the ROM and FOM physics, thereby limiting the accuracy of the ROM.

The present work proposes and develops a *new* ROM stabilization method for LTI systems that falls into the second category of methods described above. This approach can be used to stabilize ROMs constructed using *any* choice of reduced basis (e.g., POD [8], balanced truncation [16,5], proper generalized decomposition [42], among others). The key idea, motivated by the concept of full state feedback (a.k.a. pole placement) in control theory, is to change the unstable eigenvalues of a system matrix by pushing them into the stable half of the complex plane. The eigenvalues of a ROM system matrix can be modified by applying directly full state feedback (a.k.a. pole placement) algorithms from control theory [6,7,25], that is, by adding to the ROM system a linear feedback control term, and solving for the feedback matrix such that the stabilized ROM system has a desired set of eigenvalues. However, this approach can change the ROM physics, thereby making the ROM