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Finite element modeling of linear elastodynamics problems with explicit time-integration methods and linear elements with the reduced dispersion error

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ABSTRACT

We have developed two finite element techniques with reduced dispersion for linear elastodynamics that are used with explicit time-integration methods. These techniques are based on the modified integration rule for the mass and stiffness matrices and on the averaged mass matrix approaches that lead to the numerical dispersion reduction for linear finite elements. The analytical study of numerical dispersion for the new techniques is carried out in the 1-D, 2-D and 3-D cases. The numerical study of the efficiency of the dispersion reduction techniques includes the two-stage time-integration approach with the filtering stage (developed in our previous papers) that quantifies and removes spurious high-frequency oscillations from numerical results. We have found that in contrast to the standard linear elements with explicit time-integration methods and the lumped mass matrix, the finite element techniques with reduced dispersion yield more accurate results at small time increments (smaller than the stability limit) in the 2-D and 3-D cases. The recommendations for the selection of the size of time increments are suggested. The new approaches with reduced dispersion can be easily implemented into existing finite element codes and lead to significant reduction in computation time at the same accuracy compared with the standard finite element formulations.

1. Introduction

The application of the space discretization to transient acoustics or transient linear elastodynamics problems leads to a system of ordinary differential equations in time

$$\boldsymbol{M}\ddot{\boldsymbol{U}} + \boldsymbol{C}\dot{\boldsymbol{U}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{R}, \tag{1}$$

where $\boldsymbol{M}$, $\boldsymbol{C}$, $\boldsymbol{K}$ are the mass, damping, and stiffness matrices, respectively, $\boldsymbol{U}$ is the vector of the nodal displacement, $\boldsymbol{R}$ is the vector of the nodal load. Zero viscosity, $\mathbf{C} = \mathbf{0}$, is considered in the paper. Due to the space discretization, the exact solution to Eq. (1) contains the numerical dispersion error; e.g., see [1–14] and others. The space discretization error can be decreased by the use of mesh refinement. However, this procedure significantly increases computational costs. Therefore, special techniques have been developed for the reduction in the numerical dispersion error which is also related to "the pollution effect" (e.g., see [15–17] and others for the study of the pollution error). One simple and efficient technique for acoustic and elastic wave propagation problems is based on the calculation of the mass matrix $\boldsymbol{M}$ in Eq. (1) as a weighted average of the

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consistent and lumped mass matrices; see [5–8] and others. For the 1-D case and linear finite elements, this approach reduces the error in the wave velocity for harmonic waves from the second order to the fourth order of accuracy. However, for harmonic wave propagation in the 2-D and 3-D cases, these results are not valid (nevertheless, in the multidimensional case, the averaged mass matrix yields more accurate results compared with the standard mass matrix; e.g., see the numerical results in Section 3). We should also mention that the known publications on the techniques with the averaged mass matrix do not include the effect of finite time increments on the dispersion error and on the accuracy of the numerical results. As shown in the current paper, if we use the weighting coefficients for the averaged mass matrix that are independent of time increments (as in the known approaches) and if the time increments for explicit time-integration methods are close to the stability limit then there is no advantages in the use of the averaged mass matrix compared with the lumped mass matrix.

An interesting technique with implicit and explicit time-integration methods is suggested in [10] for acoustic waves in the 2-D case. It is based on the modified integration rule for the calculation of the mass and stiffness matrices for linear finite elements. In contrast to the averaged mass matrix, the use of the modified integration rule increases the accuracy for the phase velocity from the second order to the fourth order in the general multi-dimensional case of acoustic waves. However, the applicability of this technique to elastodynamics problems has not been studied. The technique in [10] has not treated spurious oscillations that may significantly destroy the accuracy of numerical results.

We should mention that the analysis of numerical dispersion estimates the numerical error for propagation of harmonic waves. In the general case of loading (boundary conditions), the estimation of the accuracy of numerical techniques with reduced dispersion is difficult due to the presence of spurious high-frequency oscillations in numerical solutions; e.g., see [3,5].

In our previous paper [18], we have described the finite element techniques with reduced dispersion for elastodynamics that are based on implicit time-integration methods with very small time increments. These techniques significantly reduce the computation time at the same accuracy compared with the standard finite element formulations with the consistent mass matrix. However, one of the disadvantages of the use of implicit time-integration methods is the necessity to solve a system of algebraic equations that can require large computational resources for a large number of degrees of freedom. In this paper we have extended the finite element techniques with reduced dispersion that can be used with explicit time-integration methods. We have considered two techniques: one of them is based on the use of the averaged mass matrix; another is based on the modified integration rule for the mass and stiffness matrices. In contrast to our paper [18], in the present paper we have also studied the effect of time increments on the dispersion error and on the accuracy of the numerical results.

The paper consists of a modification of the system of semidiscrete equations, Eq. (1), that can be used with explicit time-integration methods and the finite elements with reduced dispersion (Section 2), the analytical study of numerical dispersion for the modified integration rule and for the averaged mass matrix techniques that are used with the 1-D, 2-D and 3-D linear finite elements and the explicit time-integration method (Section 2), a short description of the two-stage time-integration technique with the filtering of spurious oscillations that is suggested in our previous papers [18–23] (Appendix), and 1-D, 2-D and 3-D numerical examples showing the efficiency of the new technique with reduced dispersion (Section 3).

2. Dispersion analysis

In this section, we will develop the averaged mass matrix technique and the modified integration rule technique that are used with explicit time-integration methods. These two techniques significantly reduce the numerical dispersion error and the computation time compared with the standard finite element formulations for linear elastodynamics. In contrast to the study of the averaged mass matrix technique and the modified integration rule technique for the scalar wave equation considered in [5,10], the analytical study of these techniques for elastodynamics problems is much more complicated due to a greater number of non-linear terms in the dispersion equation for elastodynamics and the presence of two different types of waves (compressional and shear waves). Similar to the paper [5], we will first modify Eq. (1) for the use of explicit time-integration methods (for simplicity we assume that the damping matrix is zero, $\boldsymbol{C} = \boldsymbol{0}$). Let's rewrite Eq. (1) with the diagonal (lumped) mass matrix $\boldsymbol{D}$ as follows

$$\boldsymbol{D}\dot{\boldsymbol{V}} + \boldsymbol{K}\boldsymbol{U} = \boldsymbol{R}, \qquad (2)$$

where $\boldsymbol{V}$ is the vector of nodal velocity. Relationships between the nodal displacements and velocities can be written down as (similar to those in [5,10])

$$\boldsymbol{D}\dot{\boldsymbol{U}} = \boldsymbol{M}\boldsymbol{V} \quad \text{or} \quad \boldsymbol{D}\ddot{\boldsymbol{U}} = \boldsymbol{M}\dot{\boldsymbol{V}}, \qquad (3)$$

where $\boldsymbol{M}$ is the non-diagonal mass matrix calculated by the averaged mass matrix technique (see Eq. (8) below) or by the modified integration rule technique (see Eqs. (9), (11) and (13) below). Premultiplying Eq. (2) by $\boldsymbol{M}\boldsymbol{D}^{-1}$ and inserting Eq. (3) into Eq. (2) we will get

$$\boldsymbol{D}\ddot{\boldsymbol{U}} + \boldsymbol{M}\boldsymbol{D}^{-1}\boldsymbol{K}\boldsymbol{U} = \boldsymbol{M}\boldsymbol{D}^{-1}\boldsymbol{R}. \qquad (4)$$