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Comparison of different higher order finite element schemes for the simulation of Lamb waves

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ABSTRACT

Structural Health Monitoring (SHM) applications call for both efficient and powerful numerical tools to predict the behavior of ultrasonic guided waves. When considering waves in thin-walled structures, so called Lamb waves, conventional linear or quadratic pure displacement finite elements soon reach their limits. The spatial as well as temporal discretisation, required to obtain good quality results has to be very fine. This results in enormous computational costs (computational time and memory storage requirements) when ultrasonic wave propagation problems are solved in the time domain. To resolve this issue several higher order finite element methods with polynomial degrees $p > 2$ are proposed. The objective of the current article is to develop such higher order schemes and to verify their capabilities with respect to accuracy and numerical performance. To the best of the authors' knowledge such comparison has not been reported in literature, yet. Specifically, spectral elements based on Lagarange polynomials (SEM), p-elements using the normalized integrals of the Legendre polynomials (p-FEM) and isogeometric elements utilizing non-uniform rational B-splines (NURBS, N-FEM) are discussed in this paper. By solving a two-dimensional benchmark problem, their advantages and drawbacks with respect to Lamb wave propagation are highlighted. The results of the convergence studies are then used to derive guidelines for estimating the optimal element size for a given finite element type and polynomial degree template. These findings serve the purpose to determine the optimal mesh configuration a priori and thus, save a considerable amount of computational effort. The proposed guideline is then tested on a three-dimensional structure with a conical hole showing an excellent agreement with the predicted behaviour.

1. Introduction

The application of elastic guided waves to inspect structures has a long history, and is nowadays widely employed for online monitoring purposes ([1,2]). In 1917 Horace Lamb mathematically predicted a special type of these waves occurring in thin-walled designs [3]. Named after its discoverer, Lamb waves refer to elastic perturbations propagating in elastic solid plates (or layers) with free boundaries. The direction of the displacements is both parallel to the midplane of the plate and perpendicular to it [4]. Two basic types of Lamb wave modes can be distinguished in an homogenous, isotropic plate, namely symmetric and anti-symmetric ones. For each excitation frequency a number of propagating modes exists. They correspond to the solution of the mathematical model description of Lambs problem [2]. Both modes of these waves are highly dispersive [5] and can furthermore convert into each other [6]. Despite their complex propagation characteristics there are certain properties which make them interesting for SHM applications and account for their wide spread use. Firstly, their small wavelengths, in a higher frequency range, and secondly, only a slight loss of amplitude magnitude make them very popular for online monitoring applications. Small wavelengths are required to ensure the interaction of Lamb waves with structural damages, like cracks or flaws. The geometrical attenuation is only proportional to $1/\sqrt{r}$ [7], where r is the travelled distance from the source. As a consequence Lamb wave based damage detection devices are a very attractive and a common choice for SHM systems [8].

The simulation of ultrasonic Lamb wave propagation is a highly demanding task from a computational point of view. It requires both a fine temporal [9] and spatial [10,11] discretization to capture the different wave modes. In this regard analytical methods [12], semi-analytical and wave finite element methods (SAFE, WFE) [5,13–15] offer fast and accurate results. Therefore, they are frequently used to calculate dispersion diagrams. But these methods are not able to analyse complex three-dimensional geometries and arbitrary perturbations of the waveguide. The wave propagation in structures containing failures, such as delaminations, cracks or other defects, are hard to be described analytically. Also the WFE-approach, which is more flexible as the SAFE-method, requires a periodic structure to be employable [5].

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Furthermore, these methods are numerically more expensive if the global behaviour of the structure is to be analysed, as the computational effort per degree-of-freedom is higher compared to conventional FEM [16].

An extension of semi-analytical finite element methods has been published by Gopalakrishnan et al. [17,18]. This approach can be thought of as finite element method formulated in the frequency domain. While linear wave analysis of simple geometries is shown to be solved very efficiently even for the higher order modes, non-linear effects like the contact between debonded surfaces and delaminations cannot be treated, since the problem is solved in frequency domain – Fast Fourier transform (FFT) is only viable for linear systems. Moreover, if transient time-domain solutions are wanted the calculation time increases significantly in order to avoid wrap around errors [16].

Considering problems dealing with three-dimensional, complex geometries it is in general inevitable to implement discretization techniques in all three spatial directions and also in time. When dealing with thin-walled structures a natural approach is to deploy finite shell elements to discretize the model. This type of finite elements is founded on a dimensionally reduced theory and has numerical advantages when thin-walled component parts are to be examined. Approaches of this kind have been proposed by Ostachowicz et al. [19–21] and Fritzen et al. [22,23]. Since the variation of the displacement field over the thickness of the structure is neglected, the symmetric Lamb wave modes cannot be resolved. Additionally, they have the drawback that multi-layered materials and complex three-dimensional stress states arising at welded joints or rivets, for example cannot be captured easily. Hence, approaches based on utilizing higher order shell finite elements [19,24–26] are also not suitable if all observed phenomena are to be captured.

While certain representatives of three-dimensional approaches like the local interaction simulation approach (LISA) by Lee and Staszewski [27] or the mass-spring lattice model (MSLM) by Yim and Sohn [28] are also confined to non-complex geometries, finite element methods (FEM) in general offer a broader variety of applications and are not limited to special assumptions, such as e.g. material parameters or geometrical regularities. Since the convergence rate of conventional lower order FEM formulations (h-FEM) is rather low, also when dealing with Lamb wave propagation problems, in recent years the focus of research has shifted to the implementation of high-order shape functions. In the literature a large variety of higher order shape-functions has been proposed, such as the Lagrange-polynomials on the Gauss–Lobatto-Legendre grid also referred to as *Spectral Element Method* (SEM) in the time-domain [29,21,30] (the term SEM is used according to Ostachowicz et al. [26]), the normalized integrals of the Legendre-polynomials resulting in the *hierarchical p-FEM* [31–35], and the application of Non-Uniform Rational B-Splines (NURBS) termed *N-FEM* [36–38]. Heretofore, the SEM has been used almost exclusively for high frequency wave propagation problems, and the other two mentioned approaches have been principally utilized for static problems including non-linear analyses, plasticity etc.

Since our goal is to describe the Lamb wave behavior in arbitrary geometries while confining the computational costs to a realizable extent, in our opinion only the higher order finite element methods are a recommendable choice. The mutual benefits and drawbacks of the SEM, p-FEM and N-FEM regarding the application to Lamb wave propagation problems have to the best of our knowledge not been analysed so far. Consequently the objective of this paper is to give a quantitative comparison of these methods in the time domain. The benchmark problem is a two-dimensional Lamb wave propagation in an isotropic plate. This example has been chosen as an analytical solution is given in the literature [2]. The intention of this benchmark is the characterization of the convergence properties and the numerical effort of the three proposed higher order finite element approaches. This enables the user to quantify the performance and the accuracy of these methods in analyzing Lamb wave propagation problems for SHM applications. While different spatial discretization techniques are tested, the same time integration scheme is applied to all analyzed cases, ensuring a good comparability of the results. Hence, the computational times are not directly evaluated, since the applied time-integration scheme is not necessarily best suited for each of the analysed finite element approaches. On this account, we compare the degrees-of-freedom required to achieve a certain level of accuracy, by the different approaches. In addition, the number of non-zero elements in the system matrices are examined, measuring the memory storage requirements of each method.

The paper is divided into three main parts. In the first part the main principles of the finite element method and the analysed shape function types, namely the Lagrange polynomials, the normalized integrals of the Legendre polynomials and the NURBS are introduced. Then, the model setup is described and the results of the convergence study are interpreted and discussed. To this end, optimal discretization schemes for the three higher order FE-approaches are proposed. These schemes are applied to a three-dimensional problem for verification reasons. Finally, the paper is concluded and an outlook to future research activities is given.

2. Finite element equations

The basis of the finite element developments is the variational formulation corresponding to Naviers equations, namely Hamiltons principle. It states that the motion of the system within the time interval $[t_1, t_2]$ is such, that under infinitesimal variation of the displacements the Hamiltonian action S vanishes, meaning that the motion of the system takes the path of the stationary action [39]

$$\delta S = \delta \int_{t_1}^{t_2} (L + W)\mathrm{d}t = 0. \tag{1}$$

Here L represents the Lagrangian function, and W the work done by the external forces. The Lagrangian is the sum of the kinetic energy and the potential strain energy. After some calculus and the substitution of Hookes law into Eq. (1), we obtain

$$0 = \underbrace{-\int_V \left[\rho \delta \mathbf{u}^T \ddot{\mathbf{u}} + \delta \boldsymbol{\varepsilon}^T \mathbf{C} \boldsymbol{\varepsilon}\right] \mathrm{dV}}_{\delta L} + \underbrace{\int_V \delta \mathbf{u}^T \mathbf{F}_V \,\mathrm{dV} + \int_{S_1} \delta \mathbf{u}^T \mathbf{F}_{S_1} \,\mathrm{dS}_1 + \sum_{i=1}^{n} \delta \mathbf{u}_i^T \mathbf{F}_i}_{\delta W}, \tag{2}$$

where ρ is the mass density, $\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}$ are the vectors of mechanical strains and stresses in Voigt-notation, respectively [40]. $\mathbf{C}$ denotes the elasticity matrix and $\ddot{\mathbf{u}}$ represents the acceleration vector. The displacement field $\mathbf{u}(\mathbf{x}, t)$ is approximated by the product of the space-dependent shape function matrix $\mathbf{N}(\mathbf{x})$ and a time-dependent vector of unknowns $\mathbf{U}(t)$,

$$\mathbf{u}(\mathbf{x}, t) = \mathbf{N}(\mathbf{x})\mathbf{U}(t). \tag{3}$$

The mechanical strain is defined as

$$\boldsymbol{\varepsilon} = \mathcal{D}\mathbf{N}\mathbf{U} = \mathbf{B}\mathbf{U}, \tag{4}$$

introducing the strain–displacement matrix $\mathbf{B} = \mathcal{D}\mathbf{N}$, where $\mathcal{D}$ is a linear differential operator relating strains and displacements. With the aid of Eq. (2), (4) and (3) the reasoning that Hamiltons principle has to be satisfied for all variations $\delta\mathbf{u} = \mathbf{N}\delta\mathbf{U}$ we obtain the well known system of equations, describing the motion of a body