Comput. Methods Appl. Mech. Engrg. 225–228 (2012) 128–141

Contents lists available at SciVerse ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

Unique real-variable expressions of the integral kernels in the Somigliana stress identity covering all transversely isotropic elastic materials for 3D BEM

L. Távara *, V. Mantič, J.E. Ortiz, F. París

Group of Elasticity and Strength of Materials, Universidad de Sevilla, Camino de los Descubrimientos s/n, 41092 Sevilla, Spain

A R T I C L E I N F O

Article history:
Received 1 December 2011
Received in revised form 14 March 2012
Accepted 20 March 2012
Available online 29 March 2012

Keywords:
Transversely isotropic material
Fundamental solution
Infinitesimal dislocation loop
Hypersingular boundary integral equation
Somigliana stress identity
Boundary element method

A B S T R A C T

A formulation and computational implementation of the hypersingular stress boundary integral equation for the numerical solution of three-dimensional linear elastic problems in transversely isotropic solids is developed. The formulation is based on a new closed-form real variable expression of the integral kernel S_{ijk} giving tractions originated by an infinitesimal dislocation loop, the source of singularity work-conjugated to stress tensor. This expression is valid for any combination of material properties and for any orientation of the radius vector between the source and field points. The expression is based on compact expressions of U_{ik} in terms of the Stroh eigenvalues on the plane normal to the radius vector. Performing double differentiation of U_{ik} for deducing the second derivative kernel $U_{ik,jl}$ the stress influence function of an infinitesimal dislocation loop Σ_{ijkl}^{loop} are first obtained, obtaining then the integral kernel S_{ijk}. The expressions of S_{ijk} and of the related kernels Σ_{ijkl}^{loop} and $U_{ik,jl}$ do not suffer from the difficulties of some previous expressions, obtained by other authors in different ways, with complex valued functions appearing for some combinations of material parameters and/or with division by zero for the radius vector at the rotational-symmetry axis. The expressions of the above mentioned kernels have been presented in a form suitable for an efficient computational implementation. The correctness of these expressions and of their implementation in a three-dimensional collocational BEM code has been tested numerically by solving problems with known analytic solutions for different classes of transversely isotropic materials. The obtained expressions will be useful in the development of BEM codes applied to composite materials, geomechanics and biomechanics. In particular, an application to biomechanics of the BEM code developed is shown. Additionally, these expressions can be employed in the distributed dislocation technique to solve crack problems.

1. Introduction

A key issue in the numerical solution of boundary integral equations (BIEs) of elasticity by the boundary element method (BEM) [1–5] is to evaluate the pertinent integral kernels, represented by the (displacement) fundamental solution and its derivatives, also called free-space Green's functions, in an accurate and efficient way. The present work dealing mainly with the evaluation of the hypersingular integral kernel S_{ijk} $(i,j,k = 1,2,3)$ in the 3D Somigliana stress identity for transversely isotropic materials can be considered as a continuation of a previous work of the authors [6]. Thus, for the sake of simplicity, the same notation as in [6] will be used hereinafter.

A comprehensive review of the history of the evaluation of 3D fundamental solutions and their derivatives for general anisotropic materials, and in particular for transversely isotropic materials, can be found in [6]. Thus, for the sake of brevity, only the main contributions to the development and applications of different kinds of expressions of the second-order derivative of the 3D fundamental solutions and the related integral kernels will be briefly reviewed herein.

Barnett [7] obtained integral representations of the first- and second-order derivatives of the fundamental solution for a general anisotropic material, $U_{ik}(x)$, by using Fourier transforms and suitable numerical schemes. Recently, Lee [8,9] deduced new general analytical expressions of the first- and second-order derivatives of a novel expression of $U_{ik}(x)$ introduced by Ting and Lee [10] in terms of the Stroh eigenvalues.

BEM applications to anisotropic materials started with the work of Wilson and Cruse [11], who implemented the expressions of $U_{ik}(x)$ and its first- and second-order derivatives in terms of a 1-D integral over the unit circle and achieved an efficient numerical procedure by tabulating the values of $U_{ik}(x)$ and its derivatives (with respect to spherical angles) and finally by interpolating these values in BEM calculations.

* Corresponding author. Tel.: +34 954487299; fax: +34 954461637.
E-mail addresses: ltavara@esi.us.es (L. Távara), mantic@esi.us.es (V. Mantič), jortiz@esi.us.es (J.E. Ortiz), paris@esi.us.es (F. París).

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http://dx.doi.org/10.1016/j.cma.2012.03.014

Another numerical procedure for a direct evaluation of $U_{ik}(x)$ and its derivatives, which is more accurate and more efficient (in terms of both computer storage and time), was developed by Gray and co-workers [12–14] from expressions obtained by residue calculations, covering also the degenerate cases with multiple poles. Another 3-D BEM implementation based on residue calculations was developed by Tonon et al. [15], evaluating the first- and second-order derivatives using a Lagrange interpolation scheme that results in a finite difference equation. An original alternative scheme for the integrations in 3D BEM was presented by Wang and Denda [16], where surface integrals of anisotropic Green's functions over linear triangular elements are partially evaluated analytically resulting in 1-D integrals over the unit semi-circle. Weak-form of anisotropic BIEs developed by Rungamornrat and Mear [17] allowed them to transform the hypersingular and strongly singular integrals into weakly singular ones which can be relatively easily evaluated.

BEM implementations of expressions of anisotropic fundamental solutions and their derivatives using the Stroh formalism introduced by Lee [8,9], were recently developed by Shiah et al. [18,19], Tan et al. [20] and Buroni et al. [21].

Extensions of the anisotropic elastic fundamental solutions and their derivatives to piezoelectric and magneto-electro-elastic materials can be found in Chen and Lin [22] and Buroni and Sáez [23]. The first- and second-order derivative of the fundamental solution for piezoelectric materials were evaluated numerically, from the integral expressions derived by the Fourier transform methods, in [22]. The fundamental solution and its first- and second-order derivatives were obtained for magneto-electro-elastic materials in [23] by combining extended Stroh formalism, Radon transform and Cauchy's residue theory.

In the particular case of transversely isotropic materials, it is noteworthy that closed-form expressions for the hypersingular integral kernel in the traction BIE, obtained from the second-order derivatives of $U_{ik}(x)$, were presented by Ariza and Domínguez [24]. These expressions were obtained using the previous works by Pan and Chou [25] based on the potential theory. To the best knowledge of the authors the expressions presented in [24] are the only ones presented in a fully explicit and closed-form manner for transversely isotropic materials to date.

Although the solution introduced in [25] and its modifications and derivatives [24,26] are usually used in BEM codes, this solution has several features which make somewhat cumbersome its implementation covering all the possible cases of transversely isotropic materials: (i) expressions depending on the values of $\triangle = \sqrt{C_{11}C_{33}} - C_{13} - 2C_{44}$ (positive, negative or zero), see Eqs. (2) and (3) in Ref. [6], and in particular its complex-variable character for $\triangle < 0$, which may require keeping values in the same branch when multi-valuedness arises; (ii) a loss of precision and/or a division by zero for the spherical angle $\phi = \pi$. Although the difficulty with the degeneracy problem at $\phi = \pi$ has been solved by Loloi [26], the above-mentioned features may still cause some difficulties in using this expression in further analytic deductions and in BEM development.

The aim of the present work is to deduce, and numerically test, a completely general and a closed-form real-variable expression of the strongly singular and hypersingular integral kernels in the traction BIE, valid for any transversely isotropic material obtained using the concepts of the Stroh formalism. As mentioned before, the present work can be considered as a continuation of a previous work [6].

In Section 2 the fundamental solution and its derivatives are expressed in terms of the so-called modulation functions. Section 3 briefly reviews the expression of the strongly singular integral kernel D_{ijk}, which was somehow deduced previously in [6]. This expression is obtained by differentiating the expression of $U_{ik}(x)$ introduced in [10], which upholds all its advantages. Section 4 presents new expressions for the second-order derivatives of the fundamental solution $U_{ik,jl}(x)$ and the associated solutions Σ_{ijkl}^{loop}, and the hypersingular integral kernel S_{ijk}. The formulation of the Somigliana stress identity, where $D_{ijk}(x)$ and $S_{ijk}(x)$ play the role of the integral kernels, is discussed in Section 5, where a BEM implementation of this identity is also presented. In Section 6, two numerical tests are presented, where the correctness of the expressions of $U_{ik,jl}(x)$, Σ_{ijkl}^{loop} and $S_{ijk}(x)$ and of their implementation in a BEM code is verified. Finally, an application that shows the capabilities of the BEM code and possible use of the expressions obtained is presented in Section 7.

2. Modulation functions of the fundamental solution and its derivatives

Let $x = (x_1, x_2, x_3)$ define a position vector of a point and let (r, θ, ϕ) be its spherical coordinates, see Fig. 1. The fundamental solution in displacements $U_{ik}(x)$ represents the displacement vector u_i at the field point x originated in an infinite homogeneous linear elastic medium subjected to a unit point force in the k-direction at the origin of coordinates O.

The fundamental solution and its derivatives with respect to cartesian coordinates can be expressed in terms of the so-called modulation functions in the following form:

$$U_{ik}(r,\theta,\phi) = \frac{1}{4\pi r}\widehat{U}_{ik}(\theta,\phi), \tag{1}$$

$$U_{ik,j}(r,\theta,\phi) = \frac{1}{4\pi r^2}\widehat{U}_{ik;j}(\theta,\phi), \tag{2}$$

$$U_{ik,j\ell}(r,\theta,\phi) = \frac{1}{4\pi r^3}\widehat{U}_{ik;j\ell}(\theta,\phi). \tag{3}$$

The modulation functions, denoted by a 'widehat', are independent of the radius r in spherical coordinates, being dependent only on the spherical angles θ and ϕ. However, for the sake of simplicity we write $\widehat{U}_{ik;j\ell}(x) = \widehat{U}_{ik;j\ell}(x/r) = \widehat{U}_{ik;j\ell}(\theta,\phi)$ and similarly for other modulation functions. While a comma between subscripts denotes a derivative with respect to a cartesian coordinate, a semicolon in a modulation function is used as a mnemonic to represent that this function is associated to the corresponding derivative kernel in (2) or (3). Malén [27] and others showed that

$$\widehat{U}_{ik}(\theta,\phi) = H_{ik}(\theta,\phi), \tag{4}$$

where H_{ik} is the Barnett–Lothe tensor of the Stroh formalism in anisotropic elasticity. A novel approach to deduce an expression

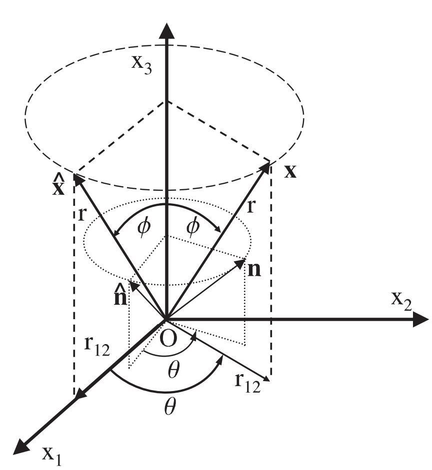

Fig. 1. Cartesian and spherical coordinates associated to a transversely isotropic material. Point x and normal vector n with the corresponding rotated point $\hat{x}$ and normal vector $\hat{n}$.