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Local time-stepping in Runge–Kutta discontinuous Galerkin finite element methods applied to the shallow-water equations

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ABSTRACT

Large geophysical flows often encompass both coarse and highly resolved regions. Approximating these flows using shock-capturing methods with explicit time stepping gives rise to a Courant–Friedrichs–Lewy (CFL) time step constraint. Even if the refined regions are sparse, they can restrict the global CFL condition to very small time steps, vastly increasing computational effort over the whole domain. One method to cope with this problem is to use locally varying time steps over the domain. These are also referred to as multi-rate methods in the ODE literature. Ideally, such methods must be conservative, accurate and easy to implement. In this study, we derive a second-order, local time stepping procedure within a Runge–Kutta discontinuous Galerkin (RKDG) framework to solve the shallow water equations. This procedure is based on previous first-order work of the second author and collaborator Kirby [1–3]. As we are interested in both coastal and overland flows due to, e.g., rainfall, wetting and drying is incorporated into the model. Numerical results are shown, which verify the accuracy and efficiency of the approach (compared to using a globally defined CFL time step), and the application of the method to rainfall–runoff scenarios.

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1. Introduction

The depth-averaged shallow water equations (SWE) are a set of hyperbolic partial differential equations (under the assumption of inviscid flow), which describe the flow of an incompressible fluid where the water depth is much smaller than the horizontal wavelength. The SWE have been extensively used to study tides, storm surges and dam breaks, among other applications. Such applications often give rise to advection-dominant flows, which are notoriously difficult to solve numerically. Furthermore, in coastal ocean applications, complex geometries, such as irregular shorelines, channels, inlets, and regions with highly varying bathymetry, must be resolved to accurately capture inland penetration of the flow. Therefore, shock-capturing methods based on unstructured finite element discretizations, such as the discontinuous Galerkin (DG) method, are often applied to the SWE. Discontinuous Galerkin methods are capable of incorporating special numerical fluxes and stability post-processing into the solution to model highly advective flows without excessive oscillations. These methods are also locally conservative, so that the continuity equation is weakly conserved element by element. Additionally, DG methods are highly parallel and allow for locally varying polynomial orders. An extensive review of DG methods can be found [4–6].

Because our interests include both coastal ocean and overland flow, there is often a natural scale separation, due to the need to resolve small channels or inlets, or, for example, when rainfall occurs. For example, in sufficiently deep, quiescent regions with no rainfall, elements may be quite large, however, overland areas with rainfall must be highly resolved, since rain may cover a relatively small portion of the total domain, and interesting runoff regions (i.e. watersheds, etc.) must be included. Efficiently handling such multi-scale problems is difficult. It is well known that for explicit time-discretization, the time step must satisfy a CFL condition to ensure numerical stability. From a global perspective, the time step calculated from the CFL is partially governed by the size of the smallest element. Element sizes in our applications may vary significantly over the domain, resulting in “over-calculations” in regions where the local CFL time step is much larger than the global CFL time step. Additionally, local spatial refinement is often required to efficiently solve for evolving fronts, density sources, etc. Upon refining, the global time step must either change as the grid is refined or be initially chosen small enough so as to ensure CFL stability to a minimum allowed element size. In either case, the global CFL is again governed by the refined region, leading to inefficient calculations. The most obvious way to efficiently deal with largely varying element sizes is to allow for a spatially varying time step, where the step-size is dependent on the locally varying CFL condition. Such methods have been previously derived and applied to conservation laws by Sanders, Dawson, Kirby and Osher [1–3,7]. This procedure was previously applied to the shallow

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water equations by Sanders [8]. In its simplest form, this procedure allows for one region of the finite element domain to have a time step M times larger than another, where M is some positive integer.

We also note the similarity of our approach with multi-rate methods and adaptive mesh refinement (AMR) methods. The AMR methods used in the GeoClaw software [9,10] have been applied to the SWE. The method uses forward Euler time stepping with time steps dictated by local CFL constraints on each refinement patch. The fluxes at the interfaces between levels are conserved in the same way described here and in [7]. The multi-rate methods described in [11] are shown to preserve second order accuracy and the TVD property. In this paper, we also focus on second order (linear) spatial approximations and second order, strong-stability-preserving time stepping in the RKDG framework. However, the method described here differs from the multi-rate approach in [11] several respects. The multi-rate method allows for time steps to vary by an integral factor (M) between regions, but requires a buffer region of size M to handle the transition. Since our domains and finite element meshes may be highly irregular and complex, we choose to essentially ignore the buffer region, while still allowing for time steps to vary by a factor of M between regions. While this is not provably second order accurate, we argue that the method is formally $\mathcal{O}(h^2 + \Delta t^2)$ away from the local-time stepping interface, where h is the mesh size and Δt is the time step. At the interface, the method reduces to $\mathcal{O}(\Delta t)$, however, the size of the interface in any practical application is also quite small; i.e., $\mathcal{O}(h)$. Therefore, while there may be some theoretical loss of accuracy due to local time stepping, our numerical results show virtually no loss of accuracy due to local time stepping, when compared to using a globally defined CFL time step.

This paper is arranged as follows. In Section 2, we present the shallow water mathematical model considered in this paper. This model includes the addition of water due to rainfall and the subsequent overland flow induced by rainfall–runoff. The numerical discretization is described in Section 3. The local time stepping method is outlined in this section. In Section 4, several numerical tests are given, which examine the accuracy of the local time stepping method. These include a verification run with an analytic solution, an idealized inlet, and three rainfall examples. The first rainfall example, known as the Iwagaki test case, is meant to validate the addition of rainfall to the model. The second two rainfall examples test the rainfall–runoff capabilities of the model with and without local time stepping.

2. Theory

The SWE are based on the three-dimensional Reynold's averaged Navier–Stokes equations for a Newtonian fluid. Averaging these equations over the vertical depth of the water, H, and applying kinematic and no-flow boundary conditions at the top and the bottom, gives rise to the conservative form of the SWE:

$$\frac{\partial H}{\partial t} + \frac{\partial (uH)}{\partial x} + \frac{\partial (vH)}{\partial y} = R(x,y,t), \tag{1}$$

$$\frac{\partial (uH)}{\partial t} + \frac{\partial \left(u^2H + \frac{1}{2}gH^2\right)}{\partial x} + \frac{\partial (uvH)}{\partial y} = gH\frac{\partial \eta}{\partial x} + \left(\tau_x^{\xi} - \tau_x^{\eta}\right) + F_x, \tag{2}$$

$$\frac{\partial (vH)}{\partial t} + \frac{\partial \left(v^2H + \frac{1}{2}gH^2\right)}{\partial y} + \frac{\partial (uvH)}{\partial x} = gH\frac{\partial \eta}{\partial y} + \left(\tau_y^{\xi} - \tau_y^{\eta}\right) + F_y, \tag{3}$$

where u and v are depth-average velocities, ξ is the water elevation relative to the geoid, $\eta = H - \xi$ is the bathymetry relative to the geoid, R is a source/sink term, which could model rainfall and/or evaporation, g is gravitational acceleration, $F_{x,y}$ are external forces (i.e. coriolis, tidal) and $\left\{\tau_{x,y}^{\xi}, \tau_{x,y}^{\eta}\right\}$ are the surface (i.e. wind, wave) and bed stresses, respectively. To arrive at these equations, a number of assumptions have been made; (1) the vertical acceleration of a fluid particle is small in comparison to the acceleration of gravity, (2) shear stresses due to the vertical velocity are small and (3) the horizontal shear terms, $\{\partial^2 u/\partial x^2, \partial^2 u/\partial y^2, \partial^2 v/\partial x^2, \partial^2 v/\partial y^2\}$ are small compared to vertical shears, $\{\partial^2 u/\partial z^2, \partial^2 v/\partial z^2\}$.

For closure, the bed stress terms must be parameterized via the depth-averaged velocities. The bed stress is often approximated by linear or quadratic functions of the velocities, however, we have used a hybrid form proposed by Westerink et al. [12], which varies the bottom-friction coefficient with the water column depth:

$$\tau_x^{\eta} = uH\left(C_f\frac{\sqrt{u^2+v^2}}{H}\right), \quad \tau_y^{\eta} = vH\left(C_f\frac{\sqrt{u^2+v^2}}{H}\right), \tag{4}$$

where

$$C_f = C_{fmin}\left(1 + \left(\frac{H_{break}}{H}\right)^{f_\theta}\right)^{f_\gamma/f_\theta}. \tag{5}$$

This formulation applies a depth-dependent, Manning-type friction law below the break depth (H_{break}) and a standard Chezy friction law when the depth is greater than the break depth. For the applications below, C_{fmin} is allowed to vary, since the bed surfaces change. Typical values for the remaining parameters are $H_{break} = 2.0$, $f_\theta = 10$ and $f_\gamma = 1.3333$.

At the water surface, stresses are induced from the impact of rain droplets. The effect is a momentum exchange between individual rain droplets and the flowing water, causing flow resistance. Zhang and Cundy [13] utilized the following expressions for surface shear stress due to rainfall:

$$\tau_x^{\xi} = -R\frac{q_x}{H} = -Ru, \tag{6}$$

$$\tau_y^{\xi} = -R\frac{q_y}{H} = -Rv. \tag{7}$$

These equations ignore rainfall splashing and are used for all the rainfall applications described in further sections.

Eqs. (1)–(3) using (4)–(7) may be cast as a single vector equation,

$$\frac{\partial w_i}{\partial t} + \nabla \cdot \mathbf{F}_i(\mathbf{w}) = s_i(\mathbf{w}), \quad \text{for } i = 1, 2, \text{ and } 3, \tag{8}$$

with vectors $\mathbf{w}$, $\mathbf{F}$ and $\mathbf{s}$ defined as

$$\begin{aligned} \mathbf{w} &= (H, uH, vH)^T, \\ \mathbf{F} = [\mathbf{f_x}, \mathbf{f_y}] &= \begin{pmatrix} uH & vH \\ u^2H + \frac{1}{2}gH^2 & uvH \\ uvH & v^2H + \frac{1}{2}gH^2 \end{pmatrix}, \\ \mathbf{s} &= \begin{pmatrix} R \\ gH\frac{\partial \xi}{\partial x} - \tau_x^{\eta} - Ru + F_x \\ gH\frac{\partial \xi}{\partial y} - \tau_y^{\eta} - Rv + F_y. \end{pmatrix}. \end{aligned} \tag{9}$$

3. Numerical methods

3.1. The discontinuous Galerkin finite element method

Hyperbolic equations often exhibit highly advective flows, which develop into sharp fronts in time. These fronts may create numerical instabilities and must be aptly dealt with. The discontinuous Galerkin method is ideally suited for such flows. Additionally, the DG method is inherently conservative, easily scalable, handles complex geometries and can easily incorporate monotonic slope limiters. An extensive review of DG methods can be found in [4,5].

Consider the hyperbolic equation,

$$\frac{\partial w}{\partial t} + \nabla \cdot \mathbf{F}(w) = s. \tag{10}$$