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Hybrid perturbation-Polynomial Chaos approaches to the random algebraic eigenvalue problem

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ABSTRACT

The analysis of structures is affected by uncertainty in the structure's material properties, geometric parameters, boundary conditions and applied loads. These uncertainties can be modelled by random variables and random fields. Amongst the various problems affected by uncertainty, the random eigenvalue problem is specially important when analyzing the dynamic behavior or the buckling of a structure. The methods that stand out in dealing with the random eigenvalue problem are the perturbation method and methods based on Monte Carlo Simulation. In the past few years, methods based on Polynomial Chaos (PC) have been developed for this problem, where each eigenvalue and eigenvector are represented by a PC expansion. In this paper four variants of a method hybridizing perturbation and PC expansion approaches are proposed and compared. The methods use Rayleigh quotient, the power method, the inverse power method and the eigenvalue equation. PC expansions of eigenvalues and eigenvectors are obtained with the proposed methods. The new methods are applied to the problem of an Euler Bernoulli beam and a thin plate with stochastic properties.

1. Introduction

The algebraic eigenvalue problem arises in a variety of fields, for example, in buckling of columns and shells [1], vibration of elastic bodies [2] and electromagnetism. Accurate methods to calculate the eigensolutions of a deterministic matrix have been available for long (see, for example, [3]), but such is not the case when the matrix considered is random. When random matrices are considered, the joint pdf of eigenvalues is only available for some special random matrix distributions, as the Gaussian Orthogonal Ensemble [4] and Wishart matrices [5]. Randomness can be introduced in the system by random parameters (e.g. Young's modulus, mass density) and consequently propagated to the system matrices such as the mass and stiffness matrices. Using the stochastic finite element method [6], these matrices in turn can be represented by a linear combination of deterministic matrices, where the coefficients are random variables [7,8].

Several methods have been developed to solve the algebraic random eigenvalue problem. Methods dealing with large amounts of uncertainty are based on Monte Carlo Simulation (MCS). These strategies are based on ordering the samples depending on the distance between them and on calculating the eigenvalues of a sample using the ones of a close sample. This ordering can be based on algorithms from the traveling salesman problem and space reduction [9], component mode synthesis [10], or can be done in a tree-type data structure [11], and the relation between eigenvalues of close samples is obtained using different initialization strategies for the power method. The start-vector used is the result from the iteration process of the previous sample. The initialization strategies and size reduction methods reduce the computational time of MCS, but for smaller uncertainties, more efficient methods are available.

Methods that can be applied to small uncertainties are based on the perturbation method [12]. First applications date from the late sixties [13,14], and a series of modified methods have been developed. A comparison of several of these methods is given by Chen et al. [15]. Other perturbation-based methods use iterations or linear combination of deterministic and first order derivative of eigenvectors to deal with larger uncertainties or to allow reanalysis of structures [16–19]. Other methods available are based on crossing theory [20], Kronecker product [21], the dimensional decomposition method [22,23], asymptotic integral method [24,25], collocation methods [26], the use of interpolations, response surface methods and meta-models [27–29] and possibilistic approaches [30]. Williams [31] used an auxiliary function where the derivative of the eigenvector equals the eigenvalue multiplied by the eigenvector.

Several authors have applied Polynomial Chaos (PC) [6] based methods to the random algebraic eigenvalue problem. A PC expansion of eigenvalues and eigenvectors was obtained by Ghosh et al. [32] using MCS for the calculation of the coefficients

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of the expansion. Verhoosel et al. [33] developed an iterative procedure based on the inverse power method and Rayleigh quotient to obtain PC expansions of the eigensolutions. Ghanem and Ghosh [34] substituted eigenvalues and eigenvectors by their PC expansion in the eigenvalue problem. Coefficients were obtained from the nonlinear problem with the help of a norm equation for the eigenvectors. A modification of the previous method using enrichment functions was derived by Ghosh and Ghanem [35].

It can be observed that even if research has been carried out both on the perturbation and PC methods for the random eigenvalue problem, no method hybridizing both approaches is yet available. Efficient methods hybridizing PC and other methods have been proposed for the elliptic problem [36,37], where a reduction of the size of the linear system to be solved was achieved. The aim of the present paper is to gain efficiency on the PC algorithms for random eigenvalue problems through the use of results from the perturbation method. The outline of the paper is as follows. The basic theories of the perturbation method and PC are discussed respectively in Sections 2.1 and 2.2. PC expansion of eigenvalues is obtained in Section 3 using the Rayleigh quotient where eigenvectors are obtained from the perturbation method or from one of the methods developed to update eigenvectors. Four new methods, namely reduced spectral power method (RSPM), reduced spectral inverse power method (RSIPM), reduced spectral constrained coefficients method (RSCCM) and spectral constrained coefficients method (SCCM) are proposed to update the eigenvectors in Section 4. The four methods allow us to obtain an updated PC expansion of the eigenvectors and eigenvalues using Rayleigh quotient. A summary of the proposed methods is given in Section 5. A comparison of the methods is performed for the problem of a beam with stochastic properties in Section 6 and for a thin plate with stochastic properties in Section 7.

2. Stochastic Finite Element method for the random eigenvalue problem

The deterministic eigenvalue problem is given by the equation

$$\mathbf{A}\mathbf{u}^{(j)} = \lambda^{(j)}\mathbf{u}^{(j)}, \tag{1}$$

where $\mathbf{A} \in \mathbb{R}^{n\times n}$ is the system matrix, $\mathbf{u}^{(j)}$ is the jth eigenvector, $\lambda^{(j)}$ is the corresponding eigenvalue and n is the degrees of freedom of the system. The system matrix is obtained from the generalized eigenvalue problem $\mathbf{K}\mathbf{y}^{(j)} = \lambda^{(j)}\mathbf{M}\mathbf{y}^{(j)}$ so that $\mathbf{A} = \mathbf{M}^{-1/2}\mathbf{K}\mathbf{M}^{-1/2}$ and $\mathbf{u}^{(j)} = \mathbf{M}^{-1/2}\mathbf{y}^{(j)}$. In a dynamic problem, matrix $\mathbf{K}$ is the stiffness matrix and $\mathbf{M}$ is the mass matrix. The system matrix $\mathbf{A}$ is assumed to be symmetric. Randomness in the matrix $\mathbf{A}$ can be introduced by a parameter (e.g. Young's modulus) represented by a random field. The random field can be approximated with a finite set of random variables using a discretization procedure (see, e.g. [7,38]). For example, the random field is discretized using the Karhunen–Loève (KL) expansion [6] and truncated after M terms. Then, the system matrix $\mathbf{A}$ can be approximated by the following KL expansion

$$\mathbf{A} = \mathbf{A}_0 + \sum_{i=1}^{M} \xi_i \mathbf{A}_i. \tag{2}$$

Here $\mathbf{A}_0$ is the mean of the system matrix and $\mathbf{A}_i$ are the matrices obtained from using the eigenfunctions of the KL expansion in the Finite Element formulation of $\mathbf{A}$. It is observed that in a more general case, the random system matrix $\mathbf{A}$ can be approximated using a set of independent identically distributed random variables $\xi_1, \ldots, \xi_M$ such that

$$\mathbf{A} = \sum_{r=1}^{P} \Gamma_r \mathbf{A}_r, \tag{3}$$

where Γ_q are a set of P polynomials of increasing order in $\xi_1, \ldots, \xi_M$ orthogonal with respect to the pdf of the random variables $\xi_1, \ldots, \xi_M$. Generally they are chosen from the Wiener–Askey scheme of polynomials [39], but can be orthogonal with respect to an arbitrary probability density function [40]. In the next subsections, the perturbation method and the PC method are used to approximate the eigenvalues and eigenvectors of the stochastic system matrix.

2.1. Perturbation method for the random eigenvalue problem

Among the various methods developed to solve the random eigenvalue problem, the perturbation method is widely used due to its simplicity and computational efficiency. The different perturbation methods available to analyze the random eigenvalue problem are based on keeping different number of terms in the Taylor series expansions. The first order perturbation of the jth eigenvalue is given by

$$\lambda^{(j)} = \lambda_0^{(j)} + \sum_{i=1}^{M} \xi_i \frac{\partial \lambda^{(j)}}{\partial \xi_i} \quad \text{where} \quad \frac{\partial \lambda^{(j)}}{\partial \xi_i} = \mathbf{u}_{j0}^T \frac{\partial \mathbf{A}}{\partial \xi_i} \mathbf{u}_{j0}. \tag{4}$$

For the case of (2), $\partial\mathbf{A}/\partial\xi_i = \mathbf{A}_i$. Perturbation methods can also be applied to eigenvectors, and the eigenvalues can then be obtained using the Rayleigh quotient. This approximation of eigenvalues is more accurate than the one obtained by directly applying the perturbation method via the Taylor series expansions [15]. If $\lambda_0^{(j)}$ and $\mathbf{u}_{j0}$ are the jth deterministic eigenvalue and the corresponding eigenvector, an expression for the first-order perturbation of the eigenvector can be given by Hasselman and Hart [14]

$$\mathbf{u}^{(j)} = \mathbf{u}_{j0} + \sum_{i=1}^{M} \xi_i \frac{\partial \mathbf{u}^{(j)}}{\partial \xi_i}. \tag{5}$$

The deterministic eigenvectors satisfy the following properties

$$\mathbf{u}_{j0}^T \mathbf{u}_{j0} = 1 \quad \text{and} \quad \mathbf{u}_{j0}^T \frac{\partial \mathbf{u}^{(j)}}{\partial \xi_i} = 0. \tag{6}$$

Different methods have been developed to calculate the derivatives of the eigenvectors. One of these methods expands the derivative of eigenvectors as a linear combination of deterministic eigenvectors [41,13], so that

$$\mathbf{u}_{ji} = \frac{\partial \mathbf{u}^{(j)}}{\partial \xi_i} = \sum_{m=1, m\neq j}^{N} \alpha_{jim} \mathbf{u}_{m0} \quad \text{where} \quad \alpha_{jim} = \frac{1}{\lambda_0^{(j)} - \lambda_0^{(m)}} \mathbf{u}_{m0}^T \frac{\partial \mathbf{A}}{\partial \xi_i} \mathbf{u}_{j0}. \tag{7}$$

For the case of (2), $\partial\mathbf{A}/\partial\xi_i = \mathbf{A}_i$. This equation is used when all deterministic eigenvectors are calculated. If only a limited number of eigenvectors were calculated, other methods described by Nelson [42] could be applied. The case of complex or repeated eigenvalues is not dealt with here. The perturbation method for such cases is derived, for example in [43–46]. For the case of repeated eigenvalues, the space corresponding to a given eigenvalue is the space spanned by its two eigenvectors, so that the proposed methods would be valid for the eigenvalues that are not repeated but not for the repeated one. The case of veering of modes is dealt with, for example in [47,48], and the proposed method does not allow to deal with this problem.

2.2. Polynomial Chaos approach for the random eigenvalue problem

Uncertainty is represented by a finite set of random variables $\{\xi_1, \ldots, \xi_M\}$ defined on the probability space $(\Xi, \mathcal{B}_\Xi, P_\xi)$. Any random quantity of interest of the system considered is then defined on this probability space, in particular, eigenvalues and eigenvectors. The eigensolutions are assumed to have finite second-order moments, and can be represented in the space of square integrable functions $\mathcal{L}^2(\Xi, dP_\xi)$, and a basis of functions Γ_k in $\mathcal{L}^2(\Xi, dP_\xi)$ can be defined. The representations of $\lambda^{(j)}$ and $\mathbf{u}^{(j)}$ on the basis functions Γ_k truncated after P terms can be given by