Comput. Methods Appl. Mech. Engrg. 217–220 (2012) 262–274

Contents lists available at SciVerse ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

Perfectly matched layers in the thin layer method

João Manuel de Oliveira Barbosa [a], Joonsang Park [b], Eduardo Kausel [c,*]

[a] Department of Civil Engineering, University of Oporto, Portugal
[b] Norwegian Geotechnical Institute, Oslo, Norway
[c] Department of Civil and Environmental Engineering, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

ARTICLE INFO

Article history:
Received 10 March 2011
Received in revised form 13 December 2011
Accepted 26 December 2011
Available online 2 January 2012

Keywords:
Perfectly matched layer method
Thin-layer method
Elastodynamics
Soil–structure interaction
Green functions

ABSTRACT

This paper explores the coupling of the perfectly matched layer technique (PML) with the thin layer method (TLM), the combination of which allows making highly efficient and accurate simulations of layered half-spaces of infinite depth subjected to arbitrary dynamic sources anywhere. It is shown that with an appropriate complex stretching of the thickness of the thin-layers, one can assemble a system of layers which fully absorbs and attenuates waves for any angle of propagation. An extensive set of numerical experiments show that the TLM + PML performance is clearly superior to that of a standard TLM model with paraxial boundaries augmented with buffer layers (TLM + PB). This finding strongly suggests that the proposed combination may in due time constitute the preferred choice for this class of problems.

1. Introduction

The thin layer method (TLM) is a semi-discrete numerical technique for the analysis of wave motion in layered media. It consists of a finite element discretization in the direction of layering combined with closed-form, analytical solutions for the remaining directions, along which the material properties are assumed to be constant. Alternatively, one can also analyze wave motion in one-dimensional waveguides of complicated cross-section—such as rails—by carrying out discretizations in not just one, but in two dimensions, and employing analytical solutions for the remaining third dimension [33], in which case the designation "thin-tube method" might be more appropriate. In general, the material layers can either be flat (i.e., horizontal layering) [17,41], or arranged into cylindrical [29] or spherical [30] layers. Fluid layers [12,26,36,39] and poroelastic layers [7] can also be considered. All of the previously cited problems belong to the more general class of partial finite elements (PFEM), in which discretizations are carried out only within some arbitrary sub-space. This class encompasses also the finite cell method [40] in which the medium is discretized in the azimuthal and meridional directions while the radial direction is handled analytically. An analysis of the dispersion characteristics of the TLM is given in [31].

Since its inception in the early 1970s [27,28,41], the TLM has found widespread use in soil dynamics and soil-structure interaction [37,38], non-destructive evaluation methods, seismic source simulations, wave propagation in waveguides of complex cross-section, wave propagation in laminated, anisotropic materials [20], waves in piezoelectric materials [8], heat diffusion in layered composites [15], consolidation in poroelastic media, solid–fluid interaction [39], and in many more areas of application. Although the origin and early development of the TLM technique hark back to the early 1970s, the designation TLM became common only since the beginning of the 1990s. Initially, the TLM was limited to bounded domains such as layers underlain by rigid base (i.e., rock) but soon paraxial boundaries (PB) became available which allowed the simulation of infinite domains [14,34,35]. A brief historical account is given in [30].

On the other hand, the perfectly matched layer (PML) is a numerical technique used for purposes similar to those of absorbing or transmitting boundaries, namely to suppress undesirable echoes and reflections of waves in infinite media modeled with discrete, finite systems. It is based on stretching the space by means of position-dependent, complex-valued scaling functions which begin with unit values at the interface or horizon delimiting the elastic region. The stretching functions then attain progressively larger complex values with distance from this horizon, which causes the waves within the PML to attenuate exponentially [16]. It can also be shown that the impedance contrast at the PML boundary is unity, in which case no reflections take place no matter what the angle of propagation of the waves entering the PML region should be.

The PML concept made its debut in the 1990s [4] and because of its excellent performance found rapid adoption in engineering science, especially for electromagnetic wave propagation models cast

* Corresponding author.
E-mail address: kausel@mit.edu (E. Kausel).

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doi:10.1016/j.cma.2011.12.006

with finite differences. In more recent years, the PML has also been used widely for problems of elastic wave propagation in both structural mechanics and in geophysics [3,9,10,13,42]. A good literature review on the subject can be found in [25].

Technical publications examining various theoretical and mathematical aspects of PMLs also abound. Of special relevance and interest to the material herein is a series of papers on the spectral properties of PMLs [5,6,11,32], which explore the characteristics of the eigenvalues of continuous PMLs—i.e., without discretization errors—in the context of electromagnetic waves in the frequency domain. It can be shown that the eigenvalues alluded to in those papers are closely related to the modes of propagation of SH (i.e., Love) waves in a layer underlain by an elastic half-space, and thus some of the findings therein are relevant to the TLM, as will be seen.

In the ensuing we apply the perfectly matched layer concept to the thin-layer method. To keep matters simple, we begin with a homogeneous stratum of complex thickness subjected to out-of-plane (i.e., anti-plane or SH) loads, define its transformation into a PML, overlay an elastic layer on top, and finally examine the characteristics and efficiency of the combination in the context of the TLM technique. We then go on exploring the more complicated case of SVP waves whose characteristics depend also on Poisson ratio. Finally, we compare the performance of the TLM + PML against that of the TLM + PB based on conventional paraxial boundaries.

2. Continuous PML for SH waves

Consider a homogeneous, elastic stratum of total depth H subjected to SH waves which propagate with celerity C_S. Following the usual strategy, we convert this stratum into a PML by transforming the vertical coordinate z into its complex, stretched counterpart $\bar{z}$ written as

$$\bar{z} = z - \mathrm{i}\Psi(z) \tag{1}$$

where $\Psi(z)$ is a function yet to be defined. The usual choice for $\Psi(z)$ guaranteeing evanescence of waves within the PML is

$$\Psi(z) = \int_0^z \psi(s)\,\mathrm{d}s \quad 0 \leqslant z \leqslant H \tag{2}$$

in which $\psi(s) > 0$ is an always positive *stretching* function. In principle the shape of $\psi(s)$ is arbitrary as long as it is continuous and $\Psi(0) < \Psi(H)$ [6]. However, once the domain is discretized into thin layers—or for that matter, into finite elements—spurious reflections take place due to the abrupt, even if small, changes in $\psi(z)$, so it behooves for this function to increase smoothly with z. A commonly used stretching function $\psi(z)$ is [10]

$$\Psi(z) = \frac{\omega_o}{\omega}\left(\frac{z}{H}\right)^m \tag{3}$$

where ω_o controls the degree of absorption of the wave and $m > 0$ defines the rate of stretching within the PML. This implies

$$\Psi(z) = \frac{\omega_o H}{\omega(m+1)}\left(\frac{z}{H}\right)^{m+1} \tag{4}$$

which can be written compactly as

$$\Psi(z) = \Omega H \zeta^{m+1} \tag{5a}$$

where

$$\Omega = \frac{\omega_0}{\omega(m+1)}, \zeta = \frac{z}{H} \tag{5b}$$

The stretched vertical coordinate then simplifies to

$$\bar{z} = z(1 - \mathrm{i}\Omega\zeta^m) \tag{6}$$

which implies a total complex depth $\overline{H} = H[1 - \mathrm{i}\Omega]$.

Consider now a plane SH wave traveling at an angle θ with respect to the vertical direction z, which we assume here to be positive downwards and starting from the free surface (Fig. 1). In stretched space, this wave can be expressed as

$$\begin{aligned} u(x,\bar{z},t) &= A\,\mathrm{e}^{\mathrm{i}\left(\omega t - x\frac{\omega}{C_S}\sin\theta - \bar{z}\frac{\omega}{C_S}\cos\theta\right)} \\ &= A\,\mathrm{e}^{\mathrm{i}\left(\omega t - x\frac{\omega}{C_S}\sin\theta - z\frac{\omega}{C_S}\cos\theta\right)}\,\mathrm{e}^{-\frac{\omega}{C_S}\cos\theta\Psi(z)} \end{aligned} \tag{7}$$

Inasmuch as (5a) guarantees $\Psi(z) > 0$ to increase monotonically and smoothly with z, and the other parameters are positive i.e., $\omega > 0$ and $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ (i.e., $\cos\theta > 0$), this expression represents an evanescent wave which decays exponentially as it propagates down. Clearly, this very same rule guarantees also that the small reflection from the bottom boundary will decay upwards, because in that case $|\theta| > \frac{1}{2}\pi$ and $\cos\theta < 0$. Now, a plane SH wave which enters the PML with an amplitude A reaches the rigid base at the bottom, $z = H$, with an amplitude $A\exp(-\omega\cos\theta\Psi(H)/C_S) = A\exp(-\omega\cos\theta\Omega H/C_S)$. In the light of Eq. (5b), this implies in turn that the total downward attenuation equals $A\exp(-\omega_0\cos\theta H/C_S/(m+1))$ which for fixed values of Ω_0H is independent of frequency. On the other hand, Eq. (5a) shows that the total stretching is controlled by the factor ΩH, and as long as this product is inversely proportional to frequency, then the total downward attenuation will remain constant. Clearly, this goal can be accomplished just as well by choosing Ω to be constant and taking the depth H of the PML to be inversely proportional to the frequency, i.e., proportional to the characteristic wavelength, as done in the ensuing. Now, since $C_S = \omega\lambda/2\pi$, with λ being the wave length, the wave reaches the base with an amplitude $A\exp(-2\pi\cos\theta\Omega H/\lambda)$. This wave elicits in turn a reflection which emerges back at the surface with an amplitude equal to the square of the previous one, i.e., $A\exp(-4\pi\cos\theta\Omega H/\lambda)$. Hence, the total roundtrip decay Δ of the wave is then

$$\Delta = \mathrm{e}^{-4\pi\Omega\eta\cos\theta} \tag{8}$$

where

$$\eta = H/\lambda \tag{9}$$

Clearly, as long as the thickness of the layer is made proportional to the wavelength (i.e., η is chosen as a constant), the effectiveness of the PML as measured by Eq. (8) for any given angle of incidence depends solely on the dimensionless parameter Ω . On the other hand, a ray entering the PML at x_0 with an inclination θ returns to the surface at a distance $r = x - x_0 = 2H\tan\theta$ from the point of penetration, i.e.,

$$\frac{r}{\lambda} = 2\eta\tan\theta \tag{10}$$

Eqs. (8)–(10) indicate that the higher the horizontal range of interest is, the higher the value needed for Ω, η, or both.

We now examine the effectiveness of this medium as a PML. For this purpose, consider an elastic half-space with shear modulus $G = 1$ Pa and shear wave velocity $C_S = 1$ m/s excited by an SH line source acting at a depth z_s with frequency $\omega = 2\pi$ rad/s. For an

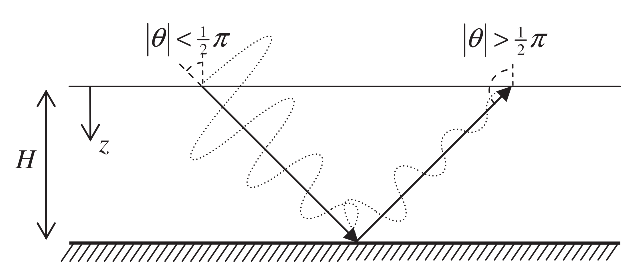

Fig. 1. Propagation of wave in the PML region.