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Equilibration techniques for solving contact problems with Coulomb friction

S. Hüeber, B. Wohlmuth *

Technische Universität München, Fakultät für Mathematik M2, Boltzmannstr. 3, 85748 Garching, Germany

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ABSTRACT

In this paper, we consider residual and equilibrated error indicators for contact problems with Coulomb friction. The contact problem is handled within the abstract framework of saddle point problems. More precisely, the non-penetration constraint and the friction law is realized as a variationally consistent weak formulation in terms of a localized dual Lagrange multiplier space. Thus from the displacement, we can easily compute in a local post-process the Lagrange multiplier which acts as a Neumann condition on the possible contact zone. Having computed the discrete Lagrange multiplier, we can apply standard error estimators by replacing the unknown Neumann data by its approximation. As it is shown in [1], this results in an error estimator for a one-sided contact problem without friction. Here, we consider more general situations and discuss two additional contact terms which measure the non-conformity of the discrete Lagrange multiplier. Numerical results in two and three dimensions illustrate the flexibility of the approach and show the influence of the material parameters on the adaptive refinement process.

1. Introduction

In many industrial applications or engineering problems, contact between deformable elastic bodies has to be considered and plays a crucial role. Although early theoretical results go back to Hertz [2], there are still many open problems. Recently a lot of research has been done on theoretical existence and uniqueness results [3,4] and on fast and robust numerical simulation techniques, see [5,6] and the references therein. From the mathematical point of view, contact problems can be analyzed within the abstract framework of variational inequalities [7–11].

In the past, penalty methods and simple node to node coupling concepts have been widely used. Nowadays they are more and more often replaced by variational consistent methods based on more sophisticated coupling strategies which also pass a suitable patch test for highly non-matching meshes. Moreover the admissibility of the solution is formulated as inequality constraint. A common approach in optimization to treat inequalities is to introduce additional variables which also have to be admissible. Then the original formulation on a convex set is equivalent to a Lagrangian approach and can be reformulated as a saddle point problem. In our situation, displacement and surface traction form a primal–dual pair of unknown variables. One set of equations reflects the equilibrium and the other one the non-penetration condition and the friction law. To obtain a stable and well-posed discrete setting, a uniform inf–sup condition has to be satisfied. Quite often such a condition is numerically verified by the Bathe–Chapelle inf–sup test, [12]. Early theoretical results on uniform stable discretization schemes for contact problems can be found in [13–15]. We refer to [16] for a first optimal a priori estimate on a two-body contact problem without friction for a biorthogonal primal–dual low order finite element pair. In the case that a vector valued Lagrange multiplier is used, there is no structural difference between a contact problem with Coulomb friction and with no friction. Thus quite often solvers and error estimators designed for contact problems without friction naturally apply as well as to contact problems with Coulomb friction. However from the theoretical point of view there is possibly a considerable difference, e.g., in the case of existence and uniqueness results, see, e.g., [3,4]. To solve the arising non-linear system many different approaches exist, e.g., interior point methods [17], SQP algorithms [18], radial return mapping or cutting plane methods [19], monotone multigrid methods [20–23] as well as penalty or augmented Lagrangian approaches [24,5,25]. Alternatively, the inequality constraints can be rewritten as a set of nondifferentiable equations, termed nonlinear complementarity (NCP-) functions (see [26–33] for some examples). Due to the lack of differentiability, the assumptions for the use of classical Newton methods [34] are not satisfied, but the so-called semi-smooth Newton methods [7,35,36] can be applied.

In this paper, we illustrate that a weakly consistent discretization based on a biorthogonal set of displacement traces and surface tractions is well suited for the numerical simulation of contact problems. It gives in the low order case optimal a priori estimates under suitable regularity assumptions, see [16,37]. Moreover the biorthogonality yields a modified but local node to node coupling concept where the simple interpolation is replaced by a stable

* Corresponding author.
E-mail address: wohlmuth@ma.tum.de (B. Wohlmuth).
URL: http://www-m2.ma.tum.de (B. Wohlmuth).

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quasi-projection. To solve the system for Coulomb friction, quite often a fixpoint approach is applied, and the problem is reduced to a sequence of simplified problems with given bound for the tangential traction. Here we apply a full Newton scheme to a scaled NCP-function [38,39]. Early results on Newton type solvers for contact problems can be found in [26–29]. We refer to the recent monograph on Lagrange multiplier methods for variational problems [40] and the references therein. It is known that semi-smooth Newton methods applied to contact problems converge locally superlinear but that global convergence is not guaranteed. The pre-asymptotic robustness can be, in particular in 3D, widely improved by a suitable local rescaling of the NCP-function and a local node-wise regularization of the Jacobian. In each step, the contact condition and its type have to be updated locally, e.g., a Robin type condition applies in the case of a sliding node. As a consequence, the semi-smooth Newton method can be implemented as a primal–dual active set strategy [36,38]. The use of active sets allows for local static condensation of either the dual variable or the corresponding primal degrees of freedom, such that only a system of the size of the displacement has to be solved in each Newton step. One of the attractive feature of this class of algorithms is that it can be easily combined with other types of non-linearities such as, e.g., plasticity and non-linear material laws. No inner and outer iteration loop due to the different types of non-linearities is required. To enhance the performance adaptive techniques based on a posteriori error estimators play an important role and are well-established for finite element methods, see [41–44] and the references therein. For abstract variational inequalities, we refer to [45–47], whereas obstacle type problems are considered in [48–51], and early approaches for contact problems can be found in [52–58].

The work is structured as follows: In Section 2, the governing equations and corresponding inequality constraints for frictional contact problems are stated and reformulated as discrete non-smooth equalities. We briefly mention the structure of the system to be solved after consistent linearization. Section 3 is devoted to different aspects of adaptive refinement. In particular, we show that the role of the coefficients of the discrete Lagrange multiplier is quite similar to the moments of equilibrated fluxes, see, e.g., [59,60]. Having worked out this link, it is quite natural to define different types of error indicators such as, e.g., residual based, equilibrated or $H(\mathrm{div})$-conforming lifted. In Section 5, we illustrate the performance of the error estimator and use it as indicator for more general situations. Although the solution of contact problems shows quite often singularities, adaptive refinement can recover the optimality of the error decay with respect to the number of nodes.

2. Problem setting and discrete formulation

In this section, we state the setting of a frictional contact problem between two elastic bodies. The two bodies in the reference configuration are given by $\Omega^{\mathrm{s}}, \Omega^{\mathrm{m}} \subset \mathbb{R}^d$, $d = 2,3$. Here the upper index s refers to the body on which the Lagrange multiplier will be defined in the discrete setting. We also refer to Ω^{s} as slave side and to Ω^{m} as master side. The boundary $\partial\Omega^k$ is assumed to be divided into three open disjoint measurable parts $\Gamma_{\mathrm{D}}^k, \Gamma_{\mathrm{N}}^k, \Gamma_C^k$ with $\mathrm{meas}(\Gamma_{\mathrm{D}}^k) > 0, k \in \{\mathrm{s},\mathrm{m}\}$. Dirichlet conditions will be set on Γ_{D}^k, Neumann data on Γ_{N}^k, and the volume forces are denoted by $\mathbf{f} \in (L^2(\Omega))^2$, $\Omega{:=}\Omega^{\mathrm{s}} \cup \Omega^{\mathrm{m}}$. On each body, we consider a homogeneous isotropic linearized Saint Venant–Kirchhoff material, where the stress tensor is given in terms of Hooke's tensor $\mathcal{C}$ by

$$\boldsymbol{\sigma}(\mathbf{v}) := \lambda \mathrm{tr}(\boldsymbol{\epsilon}(\mathbf{v}))\mathbf{Id} + 2\mu\boldsymbol{\epsilon}(\mathbf{v}) =: \mathcal{C}\boldsymbol{\epsilon}(\mathbf{v}), \tag{1}$$

and the linearized strain tensor is defined by $\boldsymbol{\varepsilon}$ $(\mathbf{v}){:=}1/2(\nabla\mathbf{v} + (\nabla\mathbf{v})^\top)$. Moreover, tr denotes the matrix trace operator and $\mathbf{Id}$ the identity in $\mathbb{R}^d$. The positive material parameters λ and μ are the Lamé parameters and are assumed to be constant on each subdomain $\Omega^k, k \in \{\mathrm{s},\mathrm{m}\}$.

The linearized elastic equilibrium condition for the displacement $\mathbf{u}{:=}(\mathbf{u}^{\mathrm{m}},\mathbf{u}^{\mathrm{s}})$ can be written as:

$$\begin{aligned} -\mathrm{div}\boldsymbol{\sigma}(\mathbf{u}) &= \mathbf{f} && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{u}_{\mathrm{D}} && \text{on } \Gamma_{\mathrm{D}} := \Gamma_{\mathrm{D}}^{\mathrm{m}} \cup \Gamma_{\mathrm{D}}^{\mathrm{s}}, \\ \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} &= \mathbf{f}_{\mathrm{N}} && \text{on } \Gamma_{\mathrm{N}} := \Gamma_{\mathrm{N}}^{\mathrm{m}} \cup \Gamma_{\mathrm{N}}^{\mathrm{s}}. \end{aligned} \tag{2}$$

Here $\mathbf{n}$ denotes the outer unit normal vector. In addition to (2), we have to satisfy the linearized non-penetration condition and the friction law. We find on Γ_C^{s} the following linearized inequality constraints for the normal components of the surface traction and the displacement

$$[u_n] \leqslant g, \quad \lambda_n \geqslant 0, \quad \lambda_n([u_n] - g) = 0, \tag{3}$$

where $\lambda_n{:=}\boldsymbol{\lambda}\mathbf{n}^{\mathrm{s}}$ is the normal component of the boundary stress $\boldsymbol{\lambda}{:=}-\boldsymbol{\sigma}(\mathbf{u}^{\mathrm{s}})\ \mathbf{n}^{\mathrm{s}}$ and $[u_n]{:=}(\mathbf{u}^{\mathrm{s}} - \mathbf{u}^{\mathrm{m}}\circ\chi)\mathbf{n}^{\mathrm{s}}$. Here, $\chi(\cdot)$ stands for a suitable mapping from Γ_C^{s} onto Γ_C^{m}, and $g(\cdot)$ is the linearized gap function between the two deformable bodies. Moreover the surface tractions on the master and slave body of the contact are in equilibrium.

In addition to (3), we have to satisfy the static Coulomb law

$$\|\boldsymbol{\lambda}_t\| \leqslant \nu\lambda_n, \quad [\mathbf{u}_t] = \alpha^2\boldsymbol{\lambda}_t, \quad \|[\mathbf{u}_t]\|(\|\boldsymbol{\lambda}_t\| - \nu\lambda_n) = 0, \tag{4}$$

where the tangential components are defined by $\boldsymbol{\lambda}_t{:=}\boldsymbol{\lambda} - \lambda_n\mathbf{n}^{\mathrm{s}}$ and $[\mathbf{u}_t]{:=}[\mathbf{u}] - [u_n]\mathbf{n}^{\mathrm{s}}$, $[\mathbf{u}]{:=}\mathbf{u}^{\mathrm{s}} - \mathbf{u}^{\mathrm{m}}\circ\chi$, $\nu \geqslant 0$ is the friction coefficient, and $\|\cdot\|$ stands for the Euclidean norm.

Remark 2.1. The special case of a contact problem between one elastic body and a rigid obstacle can be obtained in the limit case, $\lambda^{\mathrm{m}}, \mu^{\mathrm{m}} \to \infty$ and thus $\mathbf{u}^{\mathrm{m}} = \mathbf{0}$.

The discretization of the system is based on a low order pair of primal–dual variables for the displacement $\mathbf{u}$ and the surface traction $\boldsymbol{\lambda}$ on the contact zone. On each subdomain Ω^k, $k \in \{\mathrm{m},\mathrm{s}\}$, we use a family of shape regular triangulations $\mathcal{T}_l^k, l \in \mathbb{N}_0$ and set $\mathcal{T}_l := \mathcal{T}_l^{\mathrm{m}} \cup \mathcal{T}_l^{\mathrm{s}}$. The restriction of $\mathcal{T}_l^{\mathrm{m}}$ to Γ_C^{m} defines a $(d-1)$-dimensional surface mesh which will be mapped by χ^{-1} onto Γ_C^{s} resulting in possibly non-matching meshes on the contact zone. For simplicity of notation, we assume that the subdomains and the Dirichlet boundary part can be resolved by the triangulation. For the displacement, we use low order conforming finite elements and for the surface traction dual finite elements which reproduce constants

$$\begin{aligned} \mathbf{V}_l &:= \mathbf{V}_l^{\mathrm{m}} \times \mathbf{V}_l^{\mathrm{s}}, & \mathbf{V}_l^k &:= \mathrm{span}_{p\in\mathcal{P}_l^k}\{\phi_p\mathbf{e}_i, i = 1,\ldots,d\}, \\ \mathbf{M}_l &:= \mathbf{M}_l^{\mathrm{s}}, & \mathbf{M}_l^k &:= \mathrm{span}_{p\in\mathcal{P}_{C;l}^k}\{\psi_p\mathbf{e}_i, i = 1,\ldots,d\}, \end{aligned}$$

where $\mathcal{P}_l^k$ stands for all vertices of $\mathcal{T}_l^k$ not being on $\overline{\Gamma}_{\mathrm{D}}^k$, and $\mathcal{P}_{C;l}^k$ is the set of all vertices on $\overline{\Gamma}_{\mathrm{C}}^k, k \in \{\mathrm{m},\mathrm{s}\}$. Here, we assume that $\overline{\Gamma}_{\mathrm{D}}^k \cap \overline{\Gamma}_{\mathrm{C}}^k = \emptyset$. Moreover ϕ_p stands for the standard conforming nodal basis function associated with the vertex p, and ψ_p with $\mathrm{supp}\psi_p = \mathrm{supp}\phi_p|_{\Gamma_{\mathrm{C}}^k}$ satisfies the following biorthogonality relation

$$\int_{\Gamma_{\mathrm{C}}^k} \psi_p\phi_q dx = \delta_{pq}\int_{\Gamma_{\mathrm{C}}^k} \phi_q dx, \quad p,q \in \mathcal{P}_{C;l}^k. \tag{5}$$

We note that there exists no biorthogonal set of non-negative basis functions. We call a Lagrange multiplier space dual, if the set of basis functions satisfies (5). If the set of basis functions is given by the conforming nodal basis functions, we refer to it as standard Lagrange multiplier space. For $d = 2$, we can use piecewise linear but discontinuous or continuous but piecewise cubic basis functions. The weak problem formulation will be based on a suitable subset of $\mathbf{M}_l$. Let $\boldsymbol{\lambda}_l \in \mathbf{M}_l$ be given by $\boldsymbol{\lambda}_l = \sum_{p\in\mathcal{P}_{C;l}^{\mathrm{s}}}\boldsymbol{\gamma}_p\psi_p, \boldsymbol{\gamma}_p \in \mathbb{R}^d$, then we define the convex set