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Characterization of T-splines with reduced continuity order on T-meshes

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ABSTRACT

The use of T-splines [30] in Isogeometric Analysis [24] has been proposed in [5] as a tool to enhance the flexibility of isogeometric methods. If T-splines are a very general concept, their success in isogeometric analysis relies upon some basic properties that needs to be true as e.g. (i) linear independence of blending functions and (ii) polynomial reproducibility at element level.

In this paper we study these properties for T-splines of a reduced regularity order, namely, for T-splines of degree p and regularity $\alpha = p - 1 - \lfloor p/2 \rfloor$. Our results are both for odd and even degree. Under mild assumptions on the underlying T-mesh, T-splines are shown to be linearly independent and the space they span is characterized in terms of piecewise polynomials on a topological extension of the T-mesh. Also, as p is odd, we construct a new topological local refinement algorithm and demonstrate its locality properties through numerical examples.

1. Introduction

NURBS (non-uniform rational B-splines) are a standard in the Computer Aided Design community mainly because they are extremely convenient of the representation of free-form surfaces and there are very efficient algorithms to evaluate them, to refine and derefine them.

One of the main drawback of NURBS is that a NURBS surface is defined through control points which lie on a topological rectangular grid, i.e., they are constrained to a tensor product structure. This means that it may happen that many control points are superfluous and do not add information to the surface.

In geometric modeling, the most popular way to face this problem relies on the use of trimmed surfaces and the representation of a geometry with multiple NURBS patches. These techniques allows for "small" overlaps or gaps among different surface patches and the efficiency of this approach relies on the use of relative small number of control points (thousands of them at most) and on the fact that most often a global mesh is not needed.

Indeed, when using NURBS in analysis, i.e., when designing **isogeometric** methods, this approach is no more convenient: we need to work with many unknowns (millions of them, indeed) and we do need a global mesh. Isogeometric analysis has been introduced by Hughes et al. in [24] and since then it is having a growing impact in the mechanical engineering, numerical analysis and geometric modeling communities. Isogeometric analysis has been now successfully applied to several problems such as fluid dynamics [3,4,6,7,11,22], structural mechanics [2,1,8,15,20,26,32,33] and electromagnetics [12,13].

Among the various possibilities proposed in the geometric modeling literature to break the tensor product structure of NURBS, T-splines seem to be the most adapted to isogeometric analysis. T-splines has been introduced by Sederberg et al. in two pioneering papers [30,29] and are basically splines defined over meshes with T-junctions, called T-meshes. The presence of T-junctions allow for local refinement and the effort until now has been the design of suitable refinement techniques using T-splines. We refer the interested reader to the self contained paper [5] and to the very recent contributions [25,28].

This paper is about a special class of T-splines, namely the one of reduced regularity. More precisely, we assume throughout this paper that T-splines of degree p belong to C^α with $\alpha := p - 1 - \lfloor p/2 \rfloor$, where $\lfloor x \rfloor$ is the greatest integer smaller than x. Thus, C^1 cubic, C^1 quartic, C^2 quintic and so on. For this class of T-splines we address the following issues:

(i) Linear independence of T-spline blending functions.
(ii) Characterization of the function space they span in terms of piecewise polynomials.
(iii) Locality property of successive refinements.

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If linear independence is surely needed to have invertible matrices, and the locality of refinements is important for performance, we would like to point out that also characterization in terms of piecewise polynomials is fundamental. In analysis, the mesh is refined at a certain location in the domain to improve resolution. Resolution improves if the space reproduces polynomials at a finer scale. So, when using T-splines we need to make sure that refining the mesh means to have polynomials at a finer scale, otherwise, it may not improve resolution. If it is the case, the quality of the solution we compute may not improve after refinement. As shown in [10], T-splines are not always linearly independent and clearly addressing these questions requires a certain care and an understanding of the mathematical structure of T-splines over T-meshes.

Due to the simplified setting of reduced regularity, we are able to provide the following:

(i) A proof that T-splines on T-meshes with only T-junctions (i.e., without L-junctions, see [5] for details) are linearly independent.
(ii) A characterization of T-splines in terms of piecewise polynomials, when the T-mesh is moreover *regular* (see Definition 2.3) and p is odd.
(iii) A local refinement strategy when p is odd.

Our theory uses in a fundamental way the results of [17,18]. Moreover, our local refinement strategy is inspired by the one studied in [28] for splines of maximal regularity, but it is more straightforward due to the simpler structure of splines with reduced regularity.

The paper is organized as follows. In Section 2, we introduce a T-mesh and then define the inflated version of the T-mesh on which the corresponding T-spline blending functions with a proper continuity order are discussed. Also, we describe the topologically extended version of the T-mesh where piecewise polynomials are constructed. In Section 3, we show that the T-spline blending functions are linearly independent and also that under *regular* assumption on the T-mesh, the space of T-splines is characterized by piecewise polynomials on the extended T-mesh. In Section 4, using our analysis for odd degrees p, we investigate the possibility to construct a local refinement algorithm based only on topological properties of T-meshes and then propose a new local refinement algorithm of *regular* T-meshes. Finally, numerical tests demonstrating the locality property of the proposed refinement are presented in Section 5.

2. Preliminaries

In this section we prepare notation and present preliminary results. Both in one and two dimensions, we introduce and discuss properties of two kinds of spaces: the space of T-splines on a T-mesh $\mathfrak{M}$ and the space of piecewise polynomials with a certain global regularity defined on a topological extension of the T-mesh $\mathfrak{M}$ which will be defined in Section 2.2 and be called $\widetilde{\mathfrak{M}}$.

2.1. Notation and definitions in one dimension

Let $I = [0,1]$ and suppose we have a partition of I in subinterval. We denote by $\Theta = \{v_1, v_2, \ldots, v_m\}$ the collection of its nodes. Of course it holds $0 = v_1 < v_2 < \cdots < v_m = 1$. Let p be an integer and $\alpha := p - 1 - \lfloor p/2 \rfloor$. We set:

$$C(p, I) = \{u \in C^\alpha(I) : u|_{(v_i, v_{i+1})} \in \mathcal{P}_p\},$$

where $\mathcal{P}_p$ denotes the space of polynomials of degree p. First of all, we construct spline basis functions for $C(p,I)$. To this aim, we introduce the corresponding ordered knot vector

$$\Xi := \{\underbrace{s_{-\alpha}, \ldots, s_0, s_1, s_2, \ldots, s_n, s_{n+1}, \ldots, s_{n+\alpha+1}}_{n+2\alpha+2}\}, \tag{1}$$

where $0 = s_{-\alpha} = \cdots = s_0 = s_1 \leqslant s_2 \leqslant \cdots \leqslant s_n = s_{n+1} = \cdots = s_{n+\alpha+1} = 1$. The correspondence between the partition Θ and Ξ is different if p is odd or even. When p is odd, we set

$$\begin{aligned}\Xi = {}_{\mathrm{odd}}\Xi :\\ = \{\underbrace{v_1, \ldots, v_1}_{p+1 \text{ times}}, \underbrace{v_2, \ldots, v_2}_{\alpha+1 \text{ times}}, \ldots, \underbrace{v_{m-1}, \ldots, v_{m-1}}_{\alpha+1 \text{ times}}, \underbrace{v_m, \ldots, v_m}_{p+1 \text{ times}}\},\end{aligned} \tag{2}$$

with $n + 2\alpha + 2 = 2(p+1) + (m-2)(\alpha+1)$.

When p is even, we have instead

$$\begin{aligned}\Xi = {}_{\mathrm{even}}\Xi :\\ = \{\underbrace{v_1, \ldots, v_1}_{p+1 \text{ times}}, \underbrace{v_2, \ldots, v_2}_{\alpha+2 \text{ times}}, \ldots, \underbrace{v_{m-1}, \ldots, v_{m-1}}_{\alpha+2 \text{ times}}, \underbrace{v_m, \ldots, v_m}_{p+1 \text{ times}}\},\end{aligned} \tag{3}$$

with $n + 2\alpha + 2 = 2(p+1) + (m-2)(\alpha+2)$.

We can define $S(p,\Xi)$ as the space spanned by B-spline basis functions associated with Ξ (see e.g. [16]) and clearly we have $S(p,\Xi) \equiv C(p,I)$, and $S(p,\Xi) \subset C^\alpha$ since each B-spline associated with Ξ belongs to C^α.

2.2. Notation and definitions in two dimensions

We start this section with the definition of T-mesh we are going to use all along this paper. Indeed, two remarks should be made: (i) we restrict ourselves to T-meshes without L-junctions, (ii) contrary to what is made in e.g. [5], we first define the T-mesh on the parametric space and then "inflate" it to be a T-mesh in the space of indices. We prefer this way here because in case of knot repetition the definition on the index space may lead to confusion and it is a little cumbersome. To this respect, we adopt the definition given in [17].

Thus, let $\Omega = [0,1]^2 \subset \mathbb{R}^2$ be the parametric domain, a T-mesh $\mathfrak{M}$ is rectangular tiling of the domain which is basically a rectangular grid that allows T-junctions, i.e., interior vertices having three incoming edge (see [5,10,29,30] for more details). In what follows, we will denote by $\mathcal{V}(\mathfrak{M})$, $\mathcal{E}(\mathfrak{M})$, $\mathcal{F}(\mathfrak{M})$ the set of vertices, edges and faces (elements) of the T-mesh $\mathfrak{M}$, respectively. Moreover $\mathring{\mathcal{V}}(\mathfrak{M})$ and $\mathcal{V}^b(\mathfrak{M})$ stand for the interior vertices and the boundary vertices, respectively and $\mathring{\mathcal{E}}(\mathfrak{M})$ and $\mathcal{E}^b(\mathfrak{M})$ for the interior edges and the boundary edges, respectively. Note that, due to the definition of the T-mesh, the interior vertices can be of two kinds: crossing vertices, i.e., having four incoming edges, or T-junctions, i.e., having three incoming edges. We denote these two subsets as $\mathring{\mathcal{V}}_C(\mathfrak{M})$ and $\mathring{\mathcal{V}}_T(\mathfrak{M})$, respectively.

Given now a T-mesh $\mathfrak{M}$ in the *parametric space*, we are given a partition of the s axis, Θ_s and one of the t axis, Θ_t. Given the degree p, the global knot vectors Ξ_s and Ξ_t can be constructed which ensures $\alpha = p - 1 - \lfloor p/2 \rfloor$ regularity of the corresponding blending functions. Namely, following our one dimensional construction (see (2), (3)), we set:

- when p is odd, $\Xi_s := {}_{\mathrm{odd}}\Xi_s$ and $\Xi_t := {}_{\mathrm{odd}}\Xi_t$,
- when p is even, $\Xi_s := {}_{\mathrm{even}}\Xi_s$ and $\Xi_t := {}_{\mathrm{even}}\Xi_t$.

In order to adopt notation in [5], and their definition of T-spline blending functions, given a T-mesh $\mathfrak{M}$ and a degree p, the mesh $\mathfrak{M}$ can be "inflated" to become a T-mesh in *the index space* (or in *the index/parametric space* that we can use as an intermediate representation to emphasize knot repetitions) (see [5,10,19] for details). This inflation depends on the polynomial degree. The T-mesh in *the index space* is denoted by $\widehat{\mathfrak{M}} := \widehat{\mathfrak{M}}(p, \mathfrak{M})$. Examples of this inflation are given in Fig. 1.

The mesh $\widehat{\mathfrak{M}}$ is used in [5] to define anchors and then to attach blending functions to anchors. When p is odd, an anchor is