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A nonlinear POD-Galerkin reduced-order model for compressible flows taking into account rigid body motions

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ABSTRACT

The construction of a nonlinear reduced-order model for fluid–structure interaction problems is investigated in this paper for unsteady compressible flows excited by the rigid body motion of a structure. The reduction is achieved by means of a Galerkin projection of the Navier–Stokes equations on the first POD modes resulting from the proper orthogonal decomposition. In the first part of the paper, the projection technique is carried out on a purely aerodynamic case in order (i) to validate an efficient iterative technique based on an updated QR decomposition to compute the POD modes, and (ii) to discuss the merits of different correction methods introduced to improve the long-term stability of the reduced-order model. The second and most original part of the paper deals with the construction of the reduced set of equations which arise from the projection of the compressible Navier–Stokes equations formulated in a suitable moving frame representing the rigid body motion. The expressions of the resulting non-autonomous terms appearing in the reduced-order model have also been optimized to reduce the computational costs.

1. Introduction

The modelization of unsteady aeroelastic phenomena like those involved in aircraft wings or turbomachinery is very time-consuming. Such simulations cannot be performed routinely for parametric studies which are needed to evaluate the performances and to control or optimize the system. Reduced-order models with a very small number of degrees of freedom are therefore developed since several decades in the hope of being able to reproduce almost the same dynamics as the full-order system. Specific reviews in the context of compressible aerodynamics have already been proposed [22,45] but little work has been done in the case of fluid–structure interaction [4,43,63]. Another solution would be to consider the structural motion as a parameterized shape modification and to use sensitivity analysis to introduce the effects of the motion [1,33,34].

We focus here on the proper orthogonal decomposition (POD) whose principle is to determine the optimal basis to represent the system response described by a set of snapshots [37]. On the assumption that the system variables can be decomposed on the POD basis at each time instant, the projection of the equations governing the mechanical system on each POD mode produces a small set of ordinary differential equations governing the coordinates of the variables in the basis.

Since the pioneer work of Lumley [47], the proper orthogonal decomposition has been extensively used as an efficient reduction method for a wide variety of fluid dynamics systems. Three main techniques have been developed according to the equations considered to model the flow. When the flow is linear or can be linearized, the *discrete projection* is the most straightforward technique since the projection is merely performed by means of a premultiplication by the transpose of the POD basis matrix. This type of reduced-order model has been widely used for linear stability analysis in the context of aircraft [44,49] or turbomachinery applications [22,23,31,67]. The nonlinearities can be preserved in the reduced-order model with what Lucia et al. [45] called the *projection on the residual*. At each time step, the nonlinear residual is computed in the physical space with the full-order model and is then projected on the POD basis to advance in time. This technique has been used successfully to reproduce large displacements effects [4], limit-cycles oscillations [8], or shock oscillations [46,52]. The reduction is however not optimal since the aerodynamic field involving many degrees of freedom has to be reconstructed at each time instant to evaluate the residual. Recently Carlberg et al. [14] proposed an alternative formulation with three degrees of approximation to avoid the evaluation of the whole aerodynamic field. If the nonlinearity can be viewed as the action of multilinear operators like polynomials, the *Galerkin projection* technique yields an explicit nonlinear reduced-order model

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without requiring the computation of the residuals. Otherwise, the nonlinear terms are implicit and the technique has to cope with the same drawbacks as with the projection on the residual technique. The Galerkin projection approach already investigated by the author in [54] is therefore considered here. The main difficulty which is addressed in this paper lies in the formulation of the equations for compressible flows in the presence of a moving structure.

Indeed, the majority of the developments has focused on incompressible flows for which the Navier–Stokes equations are a set of quadratic partial differential equations. The Galerkin projection therefore leads to explicit nonlinear reduced-order models which have been used to reproduce turbulent structures [5,11,47], the vortex shedding process in the wake of obstacles [17,40,60] or behind a backward-facing step [18,19] and the driven cavity flow problem [15,40] for example. Recently, such reduced-order models have been applied to the control of flows [3,10,58,59,62,66] or to fluid–structure interaction problems [43].

Three difficulties arise when dealing with POD-Galerkin reduced-order models for nonlinear compressible flows around a moving structure: the first one (i) is the choice of the variables to obtain polynomial equations, the second (ii) concerns the computation of the POD modes for large snapshots databases and the last one (iii) is related to the lack of stability.

The Navier–Stokes equations usually formulated with the conservative variables for compressible flows are not quadratic and the Galerkin projection yields an inadequate implicit formulation. For isentropic flows, Rowley et al. [57] managed to derive a quadratic reduced-order model also employed in [29] to reproduce self-sustained oscillations of acoustic waves. In the general case, quadratic equations can yet be written with the judicious use of the modified primitive variables. This formulation has been introduced for nonlinear compressible flows around a *fixed* airfoil [12,39,64]. The main contribution of this paper is the extension of this formulation to solve point (i) in such a way that a *rigidly moving* structure can be taken into account in the Navier–Stokes equations while maintaining the quadratic form which is suitable for the projection.

Even with the snapshots method of Sirovich [61], the computation of the POD modes is time consuming as the snapshots become large. Solutions based on the Lanczos algorithm [24] or a parallel domain decomposition procedure [7] have been proposed, but the solution to point (ii) adopted here is based on a QR decomposition which is iteratively enriched [16,48].

The lack of stability mentioned as point (iii) is due to the discretization scheme used to approximate the fluxes, to the truncation of the POD basis, to the non-respect of certain boundary conditions or to some simplifying assumptions [19,38,50,56]. Numerous stabilization procedures have therefore been developed (see [29] for a comparison in the case of compressible flows). For autonomous systems, the proper evaluation of the initial conditions can be sufficient to reproduce accurately the limit-cycles [2]. The stability can also be enforced by modifying the dissipation operator [5,15,60,64] or by replacing the usual L^2 inner product by another one which takes into account the spatial or temporal derivatives of the snapshots [35,39,41]. More sophisticated correction techniques based on the evaluation of the reduced-order model error have recently been developed [9,40]. General calibration techniques have finally been introduced in [18,27] to determine the optimal constant, linear and/or quadratic coefficients of the reduced-order model by minimizing an error functional. This technique has been formulated in [51,66] as a linear least-squares problem.

In this paper, we consider as a preliminary work an oscillating airfoil in a nonlinear, compressible and possibly viscous flow. Such a level of modelization is indeed required to reproduce some complex aeroelastic phenomena [21] which motivate this study. Section 2 is devoted to the formulation of the POD-Galerkin reduced-order model for compressible flows governed by the Navier–Stokes equations described in a moving frame of reference with the set of modified primitive variables. The algorithm to compute the POD modes is also briefly described. In Section 3, the correction techniques used to improve the accuracy of the reduced system response are presented. A first reduced-order model of the Navier–Stokes equations is constructed in Section 4 for a fixed airfoil to validate the iterative QR decomposition algorithm adopted to compute the POD modes. Different calibration methods are also evaluated for short- and long-term time integration. Finally, a reduced-order model of the Euler equations is built in Section 5 to reproduce the motion of a shock generated by the oscillation of a moving airfoil.

2. Construction of the POD-Galerkin reduced-order model for compressible flows

2.1. Computation of the POD modes

Let $Q = \{\boldsymbol{q}^{(m)} \in H\,; m = 1,\ldots,M\}$ be a finite set of snapshots. Each snapshot is the solution of the full-order mechanical system at the time instant $t_m \in I_s = [t_0\,; t_0 + T_s]$ and is defined on the spatial domain $\Omega \subset \mathbb{R}^d$ with $d = 1$, 2 or 3 such that $\boldsymbol{q}^{(m)} = [q_1(t_m),\ldots,q_{n_\nu}(t_m)]^\top$ is a vector of n_ν squared integrable functions of space describing the aerodynamic field. The associated Hilbert space $H = (L^2(\Omega))^{n_\nu}$ is endowed for all $\boldsymbol{q}$ and $\boldsymbol{r}$ in H with the inner product

$$\langle \boldsymbol{q}, \boldsymbol{r} \rangle = \int_\Omega \sum_{k=1}^{n_\nu} q_k r_k \mathrm{d}\Omega \tag{1}$$

and the induced norm is $\|\boldsymbol{q}\|^2 = \langle \boldsymbol{q}, \boldsymbol{q} \rangle$. The previous inner product is well-defined as long as the snapshots are dimensionless since they contain different physical quantities.

The aim of the proper orthogonal decomposition is to find a subspace $S \subset H$ of low dimension q which provides the best approximation of any member of Q. Usually the snapshots are *centered* and the problem is to find the best basis to approximate the fluctuations of the snapshots $\tilde{\boldsymbol{q}}^{(m)} = \boldsymbol{q}^{(m)} - \bar{\boldsymbol{q}}$ around a mean state defined by the discrete weighted temporal average $\bar{\boldsymbol{q}} = E[\boldsymbol{q}^{(m)}] = \sum_{m=1}^{M} \alpha_m \boldsymbol{q}^{(m)}$ with $\alpha_m > 0$ and $\sum_{m=1}^{M} \alpha_m = 1$. The subspace is defined by the basis $\Phi = \{\varphi^{(j)} \in H\,; j = 1,\ldots,q\}$ so that $S = \mathrm{span}\{\varphi^{(1)},\ldots,\varphi^{(q)}\}$. Each snapshot $\boldsymbol{q}^{(m)}$ can therefore be approximated on the subspace S by the following affine decomposition on the POD modes $\varphi^{(j)}$:

$$\boldsymbol{q}^{(m)} \approx \boldsymbol{q}^{(m)}_{\mathrm{POD}} = \bar{\boldsymbol{q}} + \sum_{j=1}^{q} a_j^{(m)} \varphi^{(j)} \quad \forall m \in [\![1; M]\!]. \tag{2}$$

The optimality statement of the POD modes $\varphi^{(j)}$ and the additional constraints of orthonormality lead to the definition [42]

$$\begin{cases} \min\limits_{\varphi^{(j)} \in H} E\left[\|\tilde{\boldsymbol{q}}^{(m)} - \sum\limits_{j=1}^{q} \langle \tilde{\boldsymbol{q}}^{(m)}, \varphi^{(j)} \rangle \varphi^{(j)} \|^2 \right] \\ \text{subject to} \quad \langle \varphi^{(i)}, \varphi^{(j)} \rangle = \delta_{ij} \end{cases} \tag{3}$$

which is equivalent to the maximization of $\sum_{j=1}^{q} \langle E[\langle \tilde{\boldsymbol{q}}^{(m)}, \varphi^{(j)} \rangle \tilde{\boldsymbol{q}}^{(m)}], \varphi^{(j)} \rangle$ [54]. Introducing the linear operator R such that for all $\boldsymbol{y} \in H, R\boldsymbol{y} = E[\langle \tilde{\boldsymbol{q}}^{(m)}, \boldsymbol{y} \rangle \tilde{\boldsymbol{q}}^{(m)}]$, the optimization problem (3) finally amounts to the resolution of the eigenvalue problem $R\varphi^{(j)} = \lambda_j \varphi^{(j)}$ for each POD mode $\varphi^{(j)}$ [37,54]. This approach is called the *direct method* since the POD modes are directly computed as the solutions of the eigenproblem. The Hilbert–Schmidt operator R has $r \leqslant M$ non-null eigenvalues and eigenvectors. The eigenvalues λ_j represent the "energy" captured by each POD mode and provide an estimation of the truncation error $\epsilon_q = \sum_{j=q+1}^{r} \lambda_j$ [42].