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Review

# Non-conforming high order approximations of the elastodynamics equation

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ABSTRACT

In this paper we formulate and analyze two non-conforming high order strategies for the approximation of elastic wave problems in heterogeneous media, namely the Mortar Spectral Element Method and the Discontinuous Galerkin Spectral Element Method. Starting from a common variational formulation we make a full comparison of the two techniques from the points of view of accuracy, convergence, grid dispersion and stability.



**Contents**



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## 1. Introduction and motivations

The possibility of inferring the physical parameter distribution of the Earth's substratum, from information provided by elastic wave propagations, has increased the interest towards computational seismology. Recent developments in this scientific discipline concern with different numerical strategies as finite differences, finite elements, but the major efforts apply to spectral element methods (see [1–7]).

A motivation is that, in geophysical or industrial applications, finite difference discretizations require very large systems of equations to model realistic rock properties and uniform meshes are needed. On the other hand, when classical finite element methods are employed for treating complex geometries, it is necessary to invert the mass matrix.

The reasons for using spectral element-based approximations can be summarized in the following lines. Firstly, the flexibility in handling complex geometries, retaining the spatial exponential convergence for locally smooth solutions. Secondly, since spectral element methods are based on the weak formulation of the elastodynamics equations, they handle naturally both interface continuity and free boundary conditions, allowing very accurate resolutions of evanescent interface and surface waves (of major interest in seismology). Finally, spectral element methods retain a high level parallel structure, thus well suited for parallel computers.

However, when dealing with complex wave phenomena, such as soil–structure interaction problems or seismic response of sedimentary basins, the geometrical and polynomial flexibility is an important task for simulating correctly the wave-front field.

For this reason we consider two different non-conforming high-order techniques, namely the Mortar Spectral Element Method (MSEM) [8,9] and the Discontinuous Galerkin Spectral Element Method (DGSEM) [10–12] to simulate seismic wave propagation in heterogeneous media. In contrast to standard conforming discretizations, as Spectral Element Method (SEM) [13,14], these techniques have the further advantages that they can accommodate discontinuities, not only in the parameters, but also in the wavefield, while preserving the energy.

Depending on the involved materials it is possible to make a partition of the computational domain. Then, in each non-overlapping subregion a spectral finite element discretization is employed. The quadrilaterals/hexahedras do not have to match between neighbouring subdomains, and different spectral approximation degrees are allowed. Therefore, the continuity of the solution at the skeleton of the decomposition is imposed weakly, either by means of a Lagrange multiplier for the MSEM, or by penalizing the jumps of the displacement on the skeleton in the DGSEM.

In the present work, starting from a displacement-based weak formulation of the elastodynamics equation, we analyze stability, convergence, accuracy, dissipation and dispersion for the MSEM and DGSEM for the space discretization combined with second order time integration scheme. In particular we prove *a priori* error bounds for both the semi-discrete and fully-discrete non-conforming methods.

A similar analysis is provided in the existing literature for a slightly different Discontinuous Galerkin formulation, for dynamic linear elasticity and viscoelasticity [12,15]. In fact the above formulation involves an additional penalty term whose physical meaning is unclear. Yet, other authors refer to that analysis when discussing their Discontinuous Galerkin schemes [16,17]. Here we modify and update the results of [12] to analyze the presented DGSEM.

In the MSEM case, at the best of our knowledge, such analysis has never been carried out before in elastodynamics, but only for elliptic and parabolic equations [8,18–20].

Since we are dealing with time-dependent problems, we also take into account of the stability and dispersion property of our numerical scheme.

For wave propagation problems, the grid dispersion criterion determines the lowest number of nodes per wavelength such that the numerical solution has an acceptable level of accuracy, while the stability criterion determines the largest time step allowed for explicit time integration schemes.

A general framework to study the numerical dispersion for the SEM was developed in [21] and analyzed for the acoustic case up to polynomial approximation degree equal to three. In [22] a complete description for the elastic case is given, based on a Rayleigh quotient approximation of the eigenvalue problem characterizing the dispersion relation.

For the DGSEM, grid dispersion has been analyzed in [23,16]. In particular in [23] the dispersion and dissipation errors of the acoustic wave equation in one space dimension are derived using the flux formulation. The results include polynomial approximation degree equal to three and conjectures on the extension to higher degrees are given. Making use of the plane wave analysis, in [16] a complete description of the grid dispersion properties is carried out for both the acoustic and the elastic case.

At the best of our knowledge, for the MSEM no results are available for the grid dispersion properties regarding the elastic wave equation.

For what concerns the stability, a classical numerical approach to solve a second order initial value problem is provided by the family of the Newmark methods [24]. The Leap-Frog Finite Difference Method is a special case of that family which is second order accurate, explicit and conditionally stable, and is the most popular one used in seismic modelling [4,21,25–27]. Other schemes like Runge–Kutta or Taylor–Galerkin, are used too [17,3,7].

In this work we derive, for the Leap-Frog Method, stability bounds linking the time step with the size of the elements and the maximum wave velocity. All results obtained are compared to those obtained with the conforming SEM case.

After introducing the elastodynamics problem and its variational formulation in Section 2, we describe in Section 3 the geometrical and functional discretization of the problem within the context of non-conforming approximations. In particular we derive the Mortar and the Discontinuous Galerkin Spectral Formulations. The algebraic aspects of the two methods are then described in Section 4. Section 5 is focused on the convergence estimates while Section 6 is devoted to the grid dispersion and stability analysis, which are carried out for 2-d case. In Sections 7 and 8 we discuss the property of accuracy and convergence of the MSEM and the DGSEM, and present a geophysical application, namely the seismic response of an alluvial basin, respectively. Finally in Section 9 we report the proofs of the convergence estimates given in Section 5.

## 2. Problem formulation

Let us consider an elastic medium occupying a finite region $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with boundary $\Gamma = \partial\Omega$ and unit outward normal