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A stochastic-deterministic coupling method for continuum mechanics

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ABSTRACT

In this paper, we present a novel approach that allows to couple a deterministic continuum model with a stochastic continuum one. The coupling strategy is performed in the Arlequin framework, which is based on a volume coupling and a partition of the energy. A suitable functional space is chosen for the weak enforcement of the continuity between the two models. The choice of this space ensures that the mean of the stochastic solution equals the deterministic solution point-wise, and enforces appropriate boundary conditions on the stochastic dimension. The proof of the existence of the solution of the mixed problem is provided. The numerical strategy is also reviewed, in particular with a view at the Monte Carlo method. Finally, examples show the interest of the method, and possible strategies for use in adaptive modeling.

1. Introduction

Classical deterministic models provide global predictions that are satisfactory for many industrial applications. However, when one is interested in a very localized behavior or quantity, or when multiscale phenomena come into play, these models may not be sufficient. For instance, the limited heterogeneity of a material modeled as a continuum might have no influence on its behavior on a large scale, while the study of a local stress intensity factor would strongly depend on the local heterogeneity of the mechanical parameters. Unfortunately, for these problems, the information necessary to parameterize the relevant, very complex, models is usually not available. Stochastic methods have therefore been proposed and now appear unavoidable in multiscale modeling.

Although the use of stochastic models and methods has expanded rapidly in the last decades, the related numerical costs are still often prohibitive. Hence, the application of these methods in a complex or industrial context remains limited. An important field of research is therefore concerned with the reduction of the costs associated with the use of stochastic methods, for example by using iterative methods specially adapted to the structure of the matrices arising in the Stochastic Finite Element (FE) method [1,2], using reduced bases for the representation of random fields [3], or using special domain decomposition techniques for parallel resolution on clusters of computers [4].

The present paper proposes an alternative to these purely mathematical/numerical approaches through the coupling of two models: one deterministic and one stochastic. The general goal is that of modeling a global problem in a mean or homogeneous way where it yields sufficient accuracy, while retaining a stochastic model where needed. Hence, additional complexity is added in the model only where required, and the general approach is both more elegant and numerically cheaper than a global all-over stochastic model would be. Further, the cuts on computational costs mean that industrial applications come within reach.

The core idea for this paper, which is the choice of the operator and functional space for the coupling (Section 3.2) was proposed originally in [5]. It is here further described, in particular by adding the proof of existence and uniqueness of a solution for the mixed problem (Theorem 3.1), and showing how a Monte Carlo approach can be considered for the resolution of that problem (Section 3.4.2).

This work is closely related to two previous works in the literature [6,7]. However, in [6], the theoretical basis, which is different from the Arlequin formulation, is less general. In particular, it is only aimed at coupling a deterministic Boundary Element method with a Stochastic FE method. In the recent work [7], the authors aim at coupling two stochastic models, one continuous, and one atomistic. However, many theoretical questions are left out. In particular, the coupling is performed between realizations of the stochastic operators, while we try to describe here the coupling at the level of the stochastic operators.

In the first part of this paper, we will present each of the two models that will be used: a deterministic continuum model with constant parameters (the "classical" one), and its stochastic counterpart, where the parameter varies randomly in space and is modeled as a random field. This first section will be concluded by a brief review of the uses and limitations of each of these models,

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taken separately, and the interest of using coupling approaches. The second part will describe the main novelty of the paper, that is the description of a general coupling approach for the two models described above, in the Arlequin framework [8,9]. Finally, applications in 1D and 2D will show the efficiency and interest of the method. In particular, a first hint at the use of this approach in the context of adaptivity will be described.

2. Description of the two mono-models

In this section, we describe the two models that will be considered in this paper: a continuum scalar mechanics model, with deterministic coefficients, and the same type of model with stochastic coefficients. Some indications will also be given concerning the uses and limitations of each of these two models in physical applications. The two models that are described in this section will be referred to as *mono-models*, in opposition to the *coupled model* that will be considered in the next section. In the section of applications, we will compare extensively the solutions obtained using the coupled model with those obtained with each of the two mono-models.

2.1. The deterministic continuum mono-model

Let us consider a domain Ω of $\mathbb{R}^d$, with outgoing normal vector n and smooth boundary $\partial\Omega$, separated into Dirichlet and Neumann boundaries Γ_D and Γ_N, such that $\Gamma_D \cup \Gamma_N = \partial\Omega$, $\Gamma_D \cap \Gamma_N = \emptyset$, and $\Gamma_D \neq \emptyset$ (Fig. 1, left). We consider Poisson's equation, with a deterministic scalar parameter $\underline{K} > 0$, considered here constant, a bulk loading field $f(x)$, defined on Ω, and a surface loading field $g_n(x)$, defined on Γ_N. Supposing for notational simplicity that the Dirichlet boundary condition is homogeneous, the weak formulation for this problem reads: find $u \in \mathcal{V}$ such that

$$a(u,v) = \ell(v), \quad \forall v \in \mathcal{V}, \tag{1}$$

where $a : \mathcal{V}\times\mathcal{V} \to \mathbb{R}$ and $\ell : \mathcal{V} \to \mathbb{R}$ are defined, respectively, by $a(u,v) = \int_\Omega \underline{K}\nabla u \cdot \nabla v\, dx$, and $\ell(v) = \int_\Omega f v\, dx + \int_{\Gamma_N} g_n v\, dx$, and

$$\mathcal{V} = \{v \in \mathcal{H}^1(\Omega), v_{|\Gamma_D} = 0\}. \tag{2}$$

Endowed with the inner product $(u,v)_\mathcal{V} = \int_\Omega \nabla u \cdot \nabla v\, dx$, and associated norm $\|u\|_\mathcal{V}^2 = \int_\Omega |\nabla u|^2\, dx$, $\mathcal{V}$ is a Hilbert space. The problem (1) can be shown to have a unique solution u, for instance using Lax–Milgram theorem [10, Chapter 2]. This unique solution can be approximated, for example, by the Finite Element method.

2.2. The stochastic continuum mono-model

Let us now consider the same domain Ω, but this time with a random fluctuating mechanical parameter. Let us model this parameter by a random field $\boldsymbol{K} \in \mathcal{L}^2(\Theta, L^\infty(\Omega))$, where $(\Theta, \mathcal{F}, P)$ is a complete probability space, with Θ a set of outcomes, $\mathcal{F}$ a σ-algebra of events of Θ, and $P : \mathcal{F} \to [0,1]$ a probability measure. We additionally assume (as in [11] for example) that this field is bounded and uniformly coercive, that is to say $\exists K_{\min}$, $K_{\max} \in (0, +\infty)$, such that

$$0 < K_{\min} \leqslant \boldsymbol{K}(x) \leqslant K_{\max} < \infty, \quad \forall x \in \Omega, \quad \text{almost surely}. \tag{3}$$

The weak formulation of the corresponding stochastic boundary value problem reads: find $\boldsymbol{u} \in \mathcal{W}$ such that

$$\mathcal{A}(\boldsymbol{u}, \boldsymbol{v}) = \mathcal{L}(\boldsymbol{v}), \quad \forall \boldsymbol{v} \in \mathcal{W}, \tag{4}$$

where $\mathcal{A} : \mathcal{W}\times\mathcal{W} \to \mathbb{R}$ and $\mathcal{L} : \mathcal{W} \to \mathbb{R}$ are defined, respectively, by $\mathcal{A}(\boldsymbol{u},\boldsymbol{v}) = E\big[\int_\Omega \boldsymbol{K}\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\, dx\big]$, and $\mathcal{L}(\boldsymbol{v}) = \int_\Omega f E[\boldsymbol{v}]\, dx + \int_{\Gamma_N} g_n E[\boldsymbol{v}]\, dx$, $E[\cdot] = \int_\Theta \cdot\, dP$ denotes the mathematical expectation,

$$\mathcal{W} = \mathcal{L}^2(\Theta, \mathcal{V}) \tag{5}$$

and $\mathcal{V}$ is defined in Eq. (2). Endowed with the inner product $(\boldsymbol{u},\boldsymbol{v})_\mathcal{W} = E\big[(\boldsymbol{u},\boldsymbol{v})_\mathcal{V}\big] = E\big[\int_\Omega \nabla\boldsymbol{u}\cdot\nabla\boldsymbol{v}\, dx\big]$, and associated norm $\|\boldsymbol{u}\|_\mathcal{W}^2 = E\Big[\|\boldsymbol{u}\|_\mathcal{V}^2\Big] = E\Big[\int_\Omega |\nabla\boldsymbol{u}|^2\, dx\Big]$, $\mathcal{W}$ is a Hilbert space.

As in the previous case, using Lax–Milgram theorem, it can be proved that this problem has a unique solution $\boldsymbol{u}$ (see for instance [11]). An approximation of that solution can then be obtained, for example, by using a Stochastic FE method [12,13] or a Monte Carlo approach [14].

Remark 2.1. We assume here that the loads f and g are deterministic but this should not be seen as a restriction of the method. In particular, the mixed formulation of the next section and its numerical approximation can be developed with both the parameter $\boldsymbol{K}(x)$ and the loads modeled as random fields.

Remark 2.2. The existence and uniqueness of the solution of the above stochastic boundary value problem can also be proved with less constraining boundary conditions. In particular, the case when

$$\mathcal{W} = \left\{\mathcal{L}^2(\Theta, \mathcal{H}^1(\Omega));\ E[\boldsymbol{v}] = 0,\ \forall x \in \Gamma_D;\ \int_{\Gamma_D} \boldsymbol{v} dx = 0,\ \text{a.s.}\right\}, \tag{6}$$

still works. The homogeneous boundary condition is therefore not imposed anymore almost surely and almost everywhere. Rather, the space average of the displacement over the Dirichlet boundary cancels almost surely. This type of boundary condition is similar to what is done within the Arlequin framework in Section 3.2.

2.3. Use of the mono-models and interest of coupling approaches

The mono-model described in Section 2.1 is interesting when the material is considered on a scale at which homogenization can take place. This statement is intrinsically linked to the quantities of interest that we aim to evaluate. In particular, the estimation of the average displacement over a given area might be well evaluated using such a homogenized mono-model. On the other hand, considering local quantities with this mono-model is not adequate. For example, following the path of the tip of a fracture can probably not be performed using this model. As a general pattern, the deterministic mono-model of Section 2.1 will be appropriate for the evaluation of average quantities in macroscale problems. In that setting, the FE method can be very efficiently implemented and yields accurate results for a relatively low cost.

The stochastic mono-model tries to take into account, to some extent, the inherent heterogeneity of the material, without falling into the pits of

- really modeling the material at a smaller scale, by considering a fully different physical setting, *e.g.* polycrystalline mechanics;

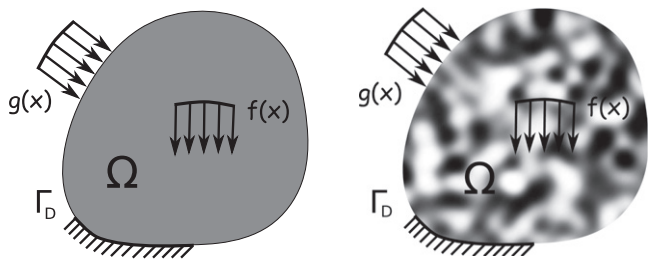

Fig. 1. Description of the two mono-models: deterministic mono-model with constant coefficient $\underline{K}$ (left) and stochastic mono-model with heterogeneous coefficient $\boldsymbol{K}(x)$ (right).