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Discontinuous Galerkin time stepping with local projection stabilization for transient convection–diffusion-reaction problems

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ABSTRACT

A time-dependent convection–diffusion-reaction problem is discretized in space by a continuous finite element method with local projection stabilization and in time by a discontinuous Galerkin method. We present error estimates for the semidiscrete problem after discretizing in space only and for the fully discrete problem. Numerical tests confirm the theoretical results.

1. Introduction

The modeling of many technical and physical processes leads to descriptions which contain time-dependent convection–diffusion-reaction equations as subproblems. Their accurate and efficient solution is often critical for accuracy and efficiency of the whole process.

There are several approaches for discretizing time-dependent convection–diffusion-reaction problems by finite element methods. Firstly, space–time elements combined with some stabilization could be used [1,2]. This results into $(d+1)$-dimensional problems in each space–time slab which are more difficult to handle than the corresponding d-dimensional problems in space. Secondly, semidiscretization as intermediate steps can be used. Here, we distinguish between horizontal and vertical methods of lines. The vertical method of lines discretizes first in space and then in time while the horizontal method of lines (or Rothe's method) applies first a time discretization which is followed by a discretization in space.

We will apply the vertical methods of lines and are interested in convection-dominated convection–diffusion-reaction problems. It is well known that standard finite element methods will lead to solutions which contain global unphysical oscillations. In order to prevent this, stabilization techniques are applied. One of the most popular methods is the streamline-upwind Petrov–Galerkin method (SUPG) which was introduced by Hughes and Brooks [3] for steady problems. However, the main drawback of the SUPG for time-dependent problems is the fact that for ensuring the consistency of the method the time derivative, the source term, and second order derivatives have to be included into the stabilization term. In particular, the assembling of the latter ones is time consuming on non-affine meshes. Moreover, the strong consistency requirement leads to a wide (and generally unphysical) coupling of the unknowns.

An alternative to SUPG are symmetric stabilization method such as the local projection stabilization (LPS) [4–6], the continuous interior penalty method (CIP) [7,8], the subgrid scale modeling (SGS) [9,10], and the orthogonal subscales method (OSS) [11,12]. They have been investigated during the last decade. The stabilization terms of CIP, OSS, and the two-level version of LPS introduce additional couplings between degrees of freedoms which do not not belong to the same finite element cell. Hence, the sparsity of the element matrices decreases and one needs appropriate data structures for an efficient implementation into a given computer code. This is not the case for the one-level variant of the LPS although the system looks larger at the first glance. However, the additional degrees of freedom which occur due to the enrichment can be eliminated locally by static condensation. In this way, one can work with the same number of degrees of freedom which are needed to achieve the appropriate approximation order. Furthermore, neither time derivatives nor second order derivatives have to be assembled for the stabilization term of LPS. Originally proposed for the Stokes problem [13], the LPS was extended

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successfully to transport problems [4]. The application of local projection methods to Oseen problems are studied in [5,6,14]. The local projection method provides additional control on the fluctuations of the gradient or parts of its. Although, the methods is weakly consistent only, the consistency error can be bounded such that the optimal order of convergence is maintained.

The discontinuous Galerkin (dG) method was first introduced by Reed and Hill [15] for neutron transport equations. The analysis of dG methods starts with the works of Lesaint and Raviart [16] and Johnson and Pitkäranta [17]. Since then, many different aspects of dG methods have been investigated in several articles. We will just mention a few of them: Delfour et al. [18], Larsson et al. [19], Schötzau and Schwab [20,21], the survey article [22], and the books [23,24].

Stabilized finite element methods for time-dependent convection–diffusion-reaction problems have been investigated by several authors. We refer to [25,26] which consider different stabilization techniques including SUPG and to [11] using OSS. The stability property of consistent stabilization methods in the small time step limit have been discussed in [27,28]. General symmetric stabilizations in space combined with the θ-method and the second order backward differentiation formula in time have been investigated in [29]. The coupling of other stabilization techniques in the one dimensional case with the finite difference method in time, in particular, vertical and horizontal methods of lines have been discussed in [30]. The standard Galerkin method in space but on a layer adapted Shishkin mesh and different time discretization have been studied in [31]. The dG method has been analyzed in space [32,33] and in space and time [34]. A numerical study of SUPG applied to time-dependent convection diffusion problems with small diffusion parameter can be found in [35]. The SUPG combined with finite differences in time for the pure transport equation has been studied in [36].

The aim of our paper is to combine the local projection stabilization in space with the discontinuous Galerkin method in time. We will give error estimates for the semidiscrete problem after discretizing in space by a finite element method with local projection stabilization and for the fully discrete problems.

The plan of the paper is as follows. Section 2 introduces the problem under consideration and defines the basic notations. The semidiscretization in space and the local projection stabilization are introduced in Section 3. Furthermore, an optimal error estimate for the semidiscretized problems will be given. Section 4 presents the error analysis for the fully discrete problem after a time discretization by a discontinuous Galerkin method. Numerical results which confirm the theoretical predictions will be shown in Section 5. Finally, Section 6 will provide some concluding remarks.

2. Notations and preliminaries

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain in $\mathbb{R}^d, d = 2, 3$, with polygonal ($d = 2$) or polyhedral ($d = 3$) Lipschitz continuous boundary $\Gamma = \partial\Omega$ and $T > 0$. We set $Q_T := \Omega \times (0,T)$ and consider the following time-dependent convection-diffusion-reaction problem:

Find $u : Q_T \to \mathbb{R}$ such that

$$\begin{cases} u_t - \varepsilon\Delta u + \mathbf{b}\cdot\nabla u + \sigma u = f & \text{in } Q_T,\\ u = 0 & \text{on } \partial\Omega\times(0,T),\\ u(\cdot,0) = u_0 & \text{in } \Omega. \end{cases} \tag{1}$$

We assume that $\mathbf{b}$, σ are independent on time t, whereas f may depend on t. Furthermore, let the data $\mathbf{b}$, σ, u_0 and f be sufficiently smooth on Ω and $\Omega \times (0,T)$, respectively. The parameter ε is supposed to be positive. By the transformation $u(x,t) = e^{Kt}v(x,t)$ with a suitably large constant K, one obtains always a system for v of form (1) such that

$$\sigma - \frac{1}{2}\,\text{div}\,\mathbf{b} \geqslant \sigma_0 > 0 \quad \text{in } \Omega. \tag{2}$$

Throughout this paper, standard notations and conventions will be used. Let $H^m(\Omega)$ denote the Sobolev space of functions with derivatives up to order m in $L^2(\Omega)$. We denote by $(\cdot,\cdot)$ the inner product in $L^2(\Omega)$ and by $\|\cdot\|$ the associated L^2-norm. The norm in $H^m(\Omega)$ is defined as

$$\|v\|_m = \left(\sum_{|\alpha|\leqslant m}\|D^\alpha v\|^2\right)^{1/2}.$$

We consider also certain Bochner spaces. For this let X be a Banach space equipped with norm $\|\cdot\|_X$ and seminorm $|\cdot|_X$. Then, we define

$$C(0,T;X) = \{v : [0,T] \to X, \quad v \text{ continuous}\},$$
$$L^2(0,T;X) = \left\{v : (0,T) \to X, \quad \int_0^T \|v(t)\|_X^2 dt < \infty\right\},$$
$$H^m(0,T;X) = \left\{v \in L^2(0,T;X) : \frac{\partial^j v}{\partial t^j} \in L^2(0,T;X), \quad 1 \leqslant j \leqslant m\right\},$$

where the derivatives $\partial^j v/\partial t^j$ are understood in the sense of distributions on $(0,T)$. In the following we use the short notation $Y(X) := Y(0,T;X)$. The norms and seminorms in the above defined spaces are given by

$$\|v\|_{C(X)} = \sup_{t\in[0,T]}\|v(t)\|_X, \quad \|v\|^2_{L^2(X)} = \int_0^T \|v(t)\|_X^2 dt,$$
$$|v|^2_{H^m(X)} = \int_0^T \left\|\frac{\partial^m v}{\partial t^m}\right\|_X^2 dt, \quad \|v\|^2_{H^m(X)} = \int_0^T \sum_{j=0}^m \left\|\frac{\partial^j v}{\partial t^j}\right\|_X^2 dt.$$

Let us introduce the space $V = H_0^1(\Omega)$, its dual space $H^{-1}(\Omega)$, and $\langle\cdot,\cdot\rangle$ for the duality product between these two spaces. Then, a function u is a weak solution of problem (1), if

$$u \in L^2\left(H_0^1\right), \quad u' \in L^2(H^{-1}), \tag{3}$$

and for almost all $t \in (0,T)$,

$$\begin{cases} \langle u'(t), v\rangle + a(u(t), v) = \langle f(t), v\rangle & \forall v \in V,\\ u(0) = u_0. \end{cases} \tag{4}$$

Here the bilinear form a is given by

$$a(u,v) := \varepsilon(\nabla u, \nabla v) + (\mathbf{b}\cdot\nabla u, v) + (\sigma u, v).$$

Note that (3) implies the continuity of u as a mapping of $[0,T] \to L^2(\Omega)$ such that the initial condition $u(0) = u_0$ is well-defined. In what follows, we shall denote by f', f'', and $f^{(q)}$ the first, second, and qth order time derivative of f, respectively.

3. Semidiscretization and local projection stabilization

For the finite element discretization of (4), let $\{\mathcal{T}_h\}$ denote a family of shape regular triangulations of Ω into d-simplices, quadrilaterals or hexahedra such that

$$\overline{\Omega} = \bigcup_{K\in\mathcal{T}_h}\overline{K}.$$

The diameter of $K \in \mathcal{T}_h$ will be denoted by h_K and the mesh size h is defined by $h := \max_{K\in\mathcal{T}_h} h_K$. We will consider the one-level LPS in which approximation and projection spaces live on the same mesh. For other variants of LPS we refer to [6,37–41].

Let $V_h \subset V$ denote the approximation space of continuous, piecewise polynomials and $\mathcal{D}_h$ be the projection space of discontinuous, piecewise polynomials. Let $\mathcal{D}_h(K) = \{q_h|_K : q_h \in \mathcal{D}_h\}$ and $\pi_K : L^2(K) \to \mathcal{D}_h(K)$ the local L^2-projection into $\mathcal{D}_h(K)$. Define the