Comput. Methods Appl. Mech. Engrg. 200 (2011) 1822–1832

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

# A finite isoclinic elasto-plasticity model with orthotropic yield function and notion of plastic spin

Manfred H. Ulz *

*Institute for Strength of Materials, Graz University of Technology, Kopernikusgasse 24/I, 8010 Graz, Austria*

ARTICLE INFO

*Article history:*
Received 21 July 2010
Received in revised form 17 November 2010
Accepted 29 January 2011
Available online 3 February 2011

*Keywords:*
Orthotropy
Isoclinic intermediate configuration
Multiplicative decomposition
Orientational evolution
Plastic spin

ABSTRACT

The paper presents a phenomenological model for the description of orthotropic elasto-plastic solids. The formulation and computational implementation are based on a multiplicative decomposition of the deformation gradient tensor into an elastic and a plastic part. This decomposition introduces an incompatible intermediate manifold, which is identified in this work as an isoclinic configuration as proposed by Mandel. A rate-independent constitutive model is developed for the modelling of isotropic elastic and orthotropic plastic material behaviour. The latter arises due to pre-existing preferred orientations in the material and is described by a Hill-type yield criterion. It is assumed that further deformations induce only a negligible change of the locally preferred orientations to each other, hence, the shape of the yield surface remains the same. Sheet forming processes are sufficiently described by such assumptions. Furthermore, the notion of a plastic spin as introduced by Dafalias, which is the spin of the continuum relative to the material substructure, plays a crucial role in the evolution of the orthotropic axes. Representative numerical simulations demonstrate the performance of the proposed model.



## 1. Introduction

Metal sheets undergo large deformations in forming processes. These sheets are normally subject to a rolling process in their production, which results in a change to the crystallographic texture and the introduction of preferred directions in the macroscopic behaviour. This gives rise to an anisotropic behaviour in plastic deformations. Moreover, these preferred directions rotate with evolution of plasticity, which is described by the notion of plastic spin. Whilst the current literature is replete with theoretical discussions regarding the nature of plastic spin, there has so far been a lack of numerical investigations into the phenomena.

In finite plastic deformations, the kinematical behaviour of the material substructure, which defines the material symmetries and hence yield loci, is not inevitably identical to that of the continuum. This idea introduced the notion of plastic spin by Dafalias [10–12] and Loret [35], which accounts for the difference of the spin of the material substructure to that of the continuum. Indeed Mandel [37,38] and Kratochvil [25,26] suggested that a complete macroscopic theory of plasticity must incorporate constitutive equations for the plastic spin as well, alongside the plastic part of the rate of deformation. Furthermore, Kratochvil [26] noted that the plastic spin vanishes in isotropic materials.

The notion of plastic spin is believed to be crucial in anisotropic materials. This has been thoroughly investigated in a series of papers with Paulun and Pecherski [43], Zbib and Aifantis [52,53], Dafalias and Aifantis [13], Aravas [2], Lubarda and Shih [36], Schieck and Stumpf [45], Kuroda [28], Levitas [32] and Itskov and Aksel [22] providing a representative sample of the work in this area. Unfortunately, the notion of plastic spin itself, has led to confusion and different interpretations in the literature, see especially van der Giessen [49] and Dafalias [14] for a summary and discussion. There is some work with a stronger focus on the numerical implementation including Dafalias [15], Han et al. [18], Choi et al. [6,7], Harrysson and Ristinmaa [19], Kim et al. [23] and Heidari et al. [20]. Next to that, there is literature about the relation of the common approach to the notion of plastic spin to Cosserat theory, see for instance Lippmann [33]. Experimental investigations into the orientational axes of metal sheets and their relation to the notion of plastic spin can be found for instance in Boehler [4], Bunge [5], Kim and Yin [24], Truong Qui and Lippmann [47] and Wu et al. [51]. A model on the mechanics of anisotropic polymers similar to this work is given in Pereda et al. [44]. Although the evolution of anisotropic directions appears to be important particularly for steel sheets, the literature is scarce on the numerical treatment of practically relevant applications such as sheet forming processes.

The model presented in this paper is based on the multiplicative decomposition of the deformation gradient $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$ according to Kröner [27] and Lee [30] into an elastic and a plastic part. This

* Tel.: +43 316 873 7166; fax: +43 316 873 7169.
*E-mail address:* manfred.ulz@tugraz.at

0045-7825/$ - see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.cma.2011.01.020

decomposition gives rise to an incompatible intermediate manifold, which is identified in this work as an isoclinic configuration as proposed by Mandel [37] and, furthermore, is the basis for the algorithmic treatment of the thermodynamic considerations. A rate-independent phenomenological constitutive model is developed to model isotropic elastic and orthotropic plastic material behaviour. The latter arises due to pre-existing preferred orientations in the material and is described by a Hill-type yield criterion. In this work it is assumed that the strains in the production process of a sheet, which induce the pre-existing preferred orientations, are much larger then the strains in the subsequent deformation, e.g., of a deep drawing process. Hence, further deformations do not change the orientation of these orthotropic axes to each other. However, these form a directional triad, which can be rotated as a whole. Furthermore, the notion of a plastic spin is introduced, which governs the evolution of the orthotropic axes. This allows the orthotropic axes to be rotated into a more favourable direction by the plastic spin. Based on these equations the algorithmic treatment yields a general return mapping algorithm including isotropic hardening effects.

The outline of the paper is as follows. Section 2 provides basic definitions and introduces the material and substructural spins, the difference of which, leads to the notion of plastic spin. Section 3 presents the elasto-plastic model employing an isoclinic intermediate configuration and provides a derivation of an orthotropic yield criterion based on representation theorems. Section 4 presents the algorithmic formulation of the model and gives the stress and consistent tangent modulus for a finite element implementation. The paper ends in Section 5 with a range of numerical examples in a finite element programme and conclusions given in Section 6.

Classical tensor notation is used throughout this paper. Fourth-order tensors are indicated with typeface $\mathbb{A}$, whereas bold face letters $\mathbf{A}$ denote second-order tensors or vectors (this should be clear from the context) and, finally, scalars are written as $A$. The expression $\mathbf{A}\otimes\mathbf{B}$ declares a dyadic product like $A_I B_J$ and $\mathbf{C}:\mathbf{D}$ denotes the contraction of tensors over two indices like $C_{IJ}D^{IJ}$.

## 2. Material versus substructural spin

### 2.1. Basic definitions

A classical continuum is considered in the following. Each particle of a body is described by the position vector $\mathbf{X}$ in a fixed reference configuration $\mathscr{B}$ in $\mathbb{R}^3$. The position vector $\mathbf{x}$ gives the same particle in the spatial configuration $\mathscr{S}$. Furthermore, the motion of the body in time $t$ is a mapping $\boldsymbol{\varphi}$, which is defined by $\mathbf{x}=\boldsymbol{\varphi}(\mathbf{X},t)$. The deformation gradient is defined as

$$\mathbf{F}=\frac{\partial\mathbf{x}}{\partial\mathbf{X}}\quad\text{with}\quad J=\det[\mathbf{F}]\geqslant 0. \tag{2.1}$$

The velocity vector $\mathbf{v}=\dot{\mathbf{x}}$ gives the definition of the spatial velocity gradient tensor as

$$\mathbf{l}=\frac{\partial\mathbf{v}}{\partial\mathbf{x}}=\dot{\mathbf{F}}\mathbf{F}^{-1}. \tag{2.2}$$

A superposed dot denotes the material time rate. The spatial velocity gradient tensor can be decomposed into a symmetric and a skew-symmetric part as

$$\mathbf{l}=\mathbf{d}+\mathbf{w},\quad\text{with}\quad\mathbf{d}=\frac{1}{2}(\mathbf{l}+\mathbf{l}^T)\quad\text{and}\quad\mathbf{w}=\frac{1}{2}(\mathbf{l}-\mathbf{l}^T). \tag{2.3}$$

The quantities $\mathbf{d}$ and $\mathbf{w}$ may be called the rate of deformation and material spin tensor, respectively.

Finally, the Jaumann co-rotational rate is introduced of a first-order tensor $\mathbf{A}$ and second-order symmetric tensor $\mathbf{B}$ as

$$\overset{\circ}{\mathbf{A}}=\dot{\mathbf{A}}-\boldsymbol{\Lambda}\mathbf{A}\quad\text{and}\quad\overset{\circ}{\mathbf{B}}=\dot{\mathbf{B}}+\mathbf{B}\boldsymbol{\Lambda}-\boldsymbol{\Lambda}\mathbf{B}, \tag{2.4}$$

where $\boldsymbol{\Lambda}$ designates a spin tensor.

### 2.2. The material spin

A unit vector $\mathbf{n}$ is considered to be attached to a material fibre. This fibre is instantaneously aligned with one of the eigenvectors of the rate of deformation tensor $\mathbf{d}$, hence, in the principal directions of stretching. The Jaumann co-rotational rate of this orientational quantity can be expressed as $\overset{\circ}{\mathbf{n}}=\dot{\mathbf{n}}-\mathbf{w}\mathbf{n}\equiv\mathbf{0}$; which equals zero, because this rate is that determined by an observer who spins with the average angular velocity of the material element. Therefore, the material spin tensor $\mathbf{w}$ at a point $\mathbf{x}$ in $\mathscr{S}$ defines the spin of a material line element and can be written as

$$\dot{\mathbf{n}}=\mathbf{w}\mathbf{n}. \tag{2.5}$$

However, it should be kept in mind that Eq. (2.5) holds only for unit vectors attached to material fibres that are momentarily aligned with the eigenvectors of $\mathbf{d}$ and not for all fibres. This allows one to interpret the material spin tensor $\mathbf{w}$ as responsible for the "macroscopic" rotation.

### 2.3. The substructural spin

A purely orientational internal variable, $\mathbf{e}$, shall be considered in an anisotropic plastic model, e.g., a unit vector specifying a fibre reinforced direction. The rate of $\mathbf{e}$ can be described as

$$\dot{\mathbf{e}}=\boldsymbol{\omega}\mathbf{e} \tag{2.6}$$

during the process of plastic deformation. In Eq. (2.6) a different spin tensor $\boldsymbol{\omega}$ is introduced on purpose. The spin $\boldsymbol{\omega}$ of the "substructure", which governs the behaviour of the internal variables, is not necessarily the same as that of the continuum $\mathbf{w}$ in the Eulerian manifold. This allows one to interpret the substructural spin tensor $\boldsymbol{\omega}$ as being responsible for the "microscopic" rotation.

### 2.4. The plastic spin

The notion of the plastic spin $\boldsymbol{\Omega}$ relates the continuum to the substructure and furthermore distinguishes between the kinematics of the continuum and the kinematics of the substructure. It is defined according to Dafalias [10–12] and Loret [35] as

$$\boldsymbol{\Omega}=\overline{\mathbf{W}}^p-\boldsymbol{\omega}. \tag{2.7}$$

The reader should note that Eq. (2.7) is defined in a plastically deformed isoclinic intermediate configuration and $\overline{\mathbf{W}}^p$ presents the plastic material spin defined therein. This is in advance of Section 3 to which the reader is referred to for details. In a complete anisotropic plastic model, a constitutive equation should be provided for the plastic spin as well. If the plastic state and thus the plastic material spin $\overline{\mathbf{W}}^p$ is known and a constitutive equation for $\boldsymbol{\Omega}$ is given, then the spin of the material substructure $\boldsymbol{\omega}$ is indirectly given, which defines the evolution of anisotropy according to Eq. (2.6).

Unfortunately, there are different definitions of the notion of plastic spin in the literature. This confusion occurred through special assumptions about the intermediate configuration for finite plasticity, see for instance van der Giessen [49] and Dafalias [14] for a discussion. This paper considers its meaning in the sense of Dafalias.

## 3. Formulation of elasto-plastic kinematics

The common description of kinematics of finite elasto-plastic deformation is based on the multiplicative decomposition of the deformation gradient $\mathbf{F}=\mathbf{F}^e\mathbf{F}^p$ as proposed by Kröner [27] and Lee