Comput. Methods Appl. Mech. Engrg. 200 (2011) 1927–1944

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

Gradient damage models: Toward full-scale computations

E. Lorentz [a,b,*], V. Godard [b]

[a] Laboratoire de Mécanique des Structures Industrielles Durables, UMR 2832 CNRS/ EDF, 1 av. Général de Gaulle, 92141 Clamart cedex, France
[b] EDF Research & Development, 1 av. Général de Gaulle, 92141 Clamart cedex, France

ARTICLE INFO

Article history:
Received 17 June 2009
Received in revised form 31 January 2010
Accepted 23 June 2010
Available online 13 July 2010

Keywords:
Damage
Strain-softening
Gradient constitutive laws
Variational formulation
Augmented lagrangian

ABSTRACT

The description of localisation and structural failure through local damage constitutive laws leads to ill-posed problems. To retrieve a well-posed problem, the material state is described through the gradients of damage in addition to the strain and the damage. Under some assumptions, it is shown that these gradient laws can be cast into a variational formulation. Its numerical treatment is reduced to the design of a mixed finite element so that algorithms such as Newton's method and path-following techniques are still available. Possible convergence difficulties related to the reduction of the damage band width and the appearance of micro instabilities are removed thanks to the design of a new class of brittle laws. Finally, numerical applications show the convergence with mesh refinement, the compatibility with coarser approaches such as Griffith's theory and cohesive crack models, the robustness in case of unstable multi-cracking, the applicability to full-scale 3D situations and the physical relevance compared to experimental data.

1. Introduction

Continuum Damage Mechanics seems an appealing approach to describe the different steps of structural failure within a single framework [1]: damage initiation, micro-crack growth, coalescence and macro-crack propagation up to the ultimate failure of the structure. However, local damage constitutive relations are known to lead to ill-posed problems caused by strain-softening, see [2] for a review, resulting in strongly mesh-dependent computations [3]. On a physical ground, damage localises in narrow bands so that the length scales of macroscopic fields and of material microstructures are no more separated, hence hindering the validity of macroscopic local laws. Nonlocal constitutive relations have been introduced in order to take into account and somewhat control the high gradients of damage. Despite some attempts to derive the nonlocal terms through homogenisation, see [4] or [5] for instance, they are mostly introduced on the basis of phenomenological or pragmatic considerations. This leads to several classes of formulations, among which:

– the regularisation of state variables, as analysed in [6], with namely the well-known "integral" nonlocal model [7] and the "implicit gradient" model [8];
– the introduction of higher order gradients of the displacement field, which encompasses in particular strain gradient models [9,10];
– the introduction of gradients of internal variable fields [11,12] or new kinematical degrees of freedom [13,14], both approaches being equivalent in quasi-static regime [15].

Following earlier work in [15,16], we focus our attention on brittle damage regularised by the introduction of damage gradient. The choice is motivated by some attractive properties: (i) the meaning of the classical stress tensor is preserved regarding the equilibrium equations, which simplifies the analysis of computational results by non-expert engineers; (ii) the regularised quantity is a field that totally controls the localisation process (no localisation can occur without high gradient of this particular field) while limiting the increase of the number of unknowns thanks to the scalar nature of damage; (iii) the damage variable is also a bounded field (on the contrary of the strain field) which should preclude issues related to a non controlled increase of the localisation band width [17]. Despite these properties, such a nonlocal formulation has seemingly received less attention in the literature than others. We think this can be attributed to the fact that the consistency condition becomes nonlocal, hence raising special numerical issues. Thus, both alternatives of the consistency condition (loading/ unloading) have been taken into account in [18] thanks to the introduction of Lagrange multipliers, in [19] by means of a mixed dual formulation and in [20] relying on a relaxation by a penalty method; but the applications stayed limited to small scale problems. In addition, it has been observed in [12,19] that the localisation band tends to narrow with increasing damage, hence ending up with mesh-dependency, a serious drawback of the approach. However, significant success have been obtained on 3D geometries

* Corresponding author. Tel.: +33 1 47 65 35 81; fax: +33 1 47 65 54 28.
E-mail address: Eric.Lorentz@edf.fr (E. Lorentz).

0045-7825/$ - see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.cma.2010.06.025

in the context of Regularised Fracture Mechanics [21] although the formulation is close to gradient damage. The main differences lie in the lack of an initial damage threshold, the presence of a residual stiffness (not a true crack, resulting in possibly high stress values) and in the relaxation of the irreversibility condition (damage is not enforced to be irreversible). These features seem questionable in the context of Damage Mechanics and deserve additional attention in order to estimate their influence.

Nevertheless encouraged by the promising results of Regularised Fracture Mechanics, we aim here at defining a gradient damage formulation that preserves the features of the local constitutive relation (namely a damage threshold, zero ultimate stiffness and irreversibility) while trying to answer the questions of mesh dependency, computational efficiency and limited intrusion in finite element codes so as to preserve the compatibility with pre-existing solution algorithms and path-following methods. This goal is achieved in three steps. First, following [16], the nonlocal constitutive relation receives a global expression in Section 2 by taking advantage of the variational properties of generalised standard materials [22,23]. Then, in Section 3, the highly nonlinear character is dealt thanks to a decomposition-coordination technique expressed on the continuous problem, before spatial discretisation. It results in the derivation of a mixed finite element, the degrees of freedom of which are the displacements, the damage and a scalar Lagrange multiplier. A symmetric consistent tangent matrix is available, hence enabling computational efficiency. At last, the choice of the damage law and especially its softening part is analysed in Section 4 by means of a 1D problem in order to get salient features with straightforward consequences on numerical robustness (convergence) and mesh-dependency. The analysis is led on the basis of a simple class of isotropic brittle law in order to focus on the difficulties related to nonlocality; in particular, distinction between tension and compression, crack closure, anisotropy and irreversible strain are not taken into account at this stage. The capabilities of the model are demonstrated through several numerical investigations in Section 5 and confronted to experimental observations in Section 6.

2. Gradient constitutive relations

2.1. The formalism of generalised standard materials at the scale of the structure

In order to benefit from variational properties for the constitutive relations, we restrict our attention to generalised standard materials, even though the resulting discrete formulation could be used beyond this framework. Moreover, the analysis is led in the context of isotropic brittle damage. In that case, the material state may be defined by the strain tensor $\boldsymbol{\varepsilon}$ and a scalar damage variable a. The damage gradient ∇a is also introduced in the constitutive law in order to control the localisation by coupling the behaviour of neighbour material points. But as explained in [15,16], the value and the gradient of the damage field a cannot be considered as independent variables since their evolution is constrained by a nonlocal relation (being the value and the gradient of a single field). Therefore, the framework of generalised standard materials is extended to the structure scale, as done in [23]: the free energy $\mathcal{F}$ and the dissipation potential $\mathcal{D}$ of the structure are defined as functionals of the strain and the damage fields. We propose the following expressions:

$$\mathcal{F}(\boldsymbol{\varepsilon}, a) = \int_\Omega \Phi(\boldsymbol{\varepsilon}, a) d\Omega + \int_\Omega \frac{c}{2} (\nabla a)^2 d\Omega, \tag{1}$$

$$\mathcal{D}(\dot{a}) = \int_\Omega \Psi(\dot{a}) d\Omega, \tag{2}$$

where Ω denotes the body domain and a dot stands for time differentiation. Φ is the local free energy; it should be a convex function with respect to $\boldsymbol{\varepsilon}$ and a separately. Ψ is the local dissipation potential which should also be convex and minimal in 0. And $c > 0$ is a parameter that controls indirectly the localisation band width through the influence of damage gradients on the free energy.

At this stage, the global constitutive relations are derived from these potentials:

$$\boldsymbol{\sigma} = \frac{\partial \mathcal{F}}{\partial \boldsymbol{\varepsilon}} (\boldsymbol{\varepsilon}, a), \tag{3}$$

$$\mathcal{Y} = -\frac{\partial \mathcal{F}}{\partial a} (\boldsymbol{\varepsilon}, a), \tag{4}$$

$$\mathcal{Y} \in \partial \mathcal{D}(\dot{a}). \tag{5}$$

$\boldsymbol{\sigma}$ and $\mathcal{Y}$ denote the stress and the driving force associated to damage. Mathematically, they are linear forms operating respectively on the strain and damage fields. $\partial \mathcal{D}$ is the sub-gradient of $\mathcal{D}$, an extension of the notion of derivative for convex non-smooth functions. It is necessary in the case of rate-independent constitutive laws where the dissipation potential is positive homogeneous of degree 1 with respect to $\dot{a}$, hence generally not differentiable. For the sake of completeness, its definition is recalled here (where the brackets stand for the duality product), see for instance [24] for complementary propositions and related properties:

$$\mathcal{Y} \in \partial \mathcal{D}(\dot{a}) \iff \quad \forall \delta a \ \mathcal{D}(\delta a) \geqslant \mathcal{D}(\dot{a}) + \langle \mathcal{Y} | \delta a - \dot{a} \rangle. \tag{6}$$

Even though the constitutive relations are totally defined through (3)–(5), this stays nevertheless quite formal and does not provide practical expressions. We will show in the next parts how these relations can be expressed first in a pointwise manner, then as a minimisation problem.

2.2. Pointwise interpretation of the gradient damage model

In the stress–strain relation (3), the strain field is involved only in a pointwise manner. Therefore, application of the linear form $\boldsymbol{\sigma}$ to any virtual strain field $\delta \boldsymbol{\varepsilon}$ simply leads to the usual expression of the internal virtual work:

$$\langle \boldsymbol{\sigma} | \delta \boldsymbol{\varepsilon} \rangle = \left\langle \frac{\partial \mathcal{F}}{\partial \boldsymbol{\varepsilon}} \middle| \delta \boldsymbol{\varepsilon} \right\rangle = \int_\Omega \frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}} : \delta \boldsymbol{\varepsilon} dx. \tag{7}$$

Thus, the equilibrium equations and the stress retain their usual local meaning, where the stress field is given by:

$$\boldsymbol{\sigma} = \frac{\partial \Phi}{\partial \boldsymbol{\varepsilon}} (\boldsymbol{\varepsilon}, a). \tag{8}$$

Note that for the sake of simplicity, the notations are chosen identical for linear forms, corresponding fields (when possible through the theorem of representation) and pointwise values (explicit reference to the position $\mathbf{x}$ is omitted).

In the combination of the state Eq. (4) and the evolution Eq. (5), the damage field is involved not only through its value but also through its gradient. On the contrary to what has been done for the stress, it is not possible to identify the driving force $\mathcal{Y}$ (linear form) with a single field Y. An application of the rules of the calculus of variations leads to the following expression where δa denotes a virtual damage field:

$$\langle \mathcal{Y} | \delta a \rangle = \int_\Omega -\frac{\partial \Phi}{\partial a} (\boldsymbol{\varepsilon}, a) \delta a \, d\Omega + \int_\Omega -c \nabla a \cdot \nabla \delta a \, d\Omega. \tag{9}$$

It is then possible to define a couple field $(Y, \mathbf{Y})$ of (scalar and vector) driving forces:

$$Y = -\frac{\partial \Phi}{\partial a} (\boldsymbol{\varepsilon}, a); \quad \mathbf{Y} = -c \nabla a. \tag{10}$$