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Guaranteed and robust a posteriori error estimates and balancing discretization and linearization errors for monotone nonlinear problems ☆

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ABSTRACT

We derive a posteriori error estimates for a class of second-order monotone quasi-linear diffusion-type problems approximated by piecewise affine, continuous finite elements. Our estimates yield a guaranteed and fully computable upper bound on the error measured by the dual norm of the residual, as well as a global error lower bound, up to a generic constant independent of the nonlinear operator. They are thus fully robust with respect to the nonlinearity, thanks to the choice of the error measure. They are also locally efficient, albeit in a different norm, and hence suitable for adaptive mesh refinement. Moreover, they allow to distinguish, estimate separately, and compare the discretization and linearization errors. Hence, the iterative (Newton–Raphson, fixed point) linearization can be stopped whenever the linearization error drops to the level at which it does not affect significantly the overall error. This can lead to important computational savings, as performing an excessive number of unnecessary linearization iterations can be avoided. A strategy combining the linearization stopping criterion and adaptive mesh refinement is proposed and numerically tested for the p-Laplacian.

1. Introduction

Let Ω be an open polyhedron of $\mathbb{R}^d$, $d \geqslant 2$. We consider the nonlinear problem in conservative form

$$-\nabla\cdot\boldsymbol{\sigma}(\nabla u) = f \quad \text{in } \Omega, \tag{1.1a}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{1.1b}$$

The scalar-valued unknown function u is termed the *potential*, and the $\mathbb{R}^d$-valued function $-\boldsymbol{\sigma}(\nabla u)$ is termed the *flux*. We assume that the flux function $\boldsymbol{\sigma} : \mathbb{R}^d \to \mathbb{R}^d$ takes the following quasi-linear form

$$\forall \xi \in \mathbb{R}^d, \quad \boldsymbol{\sigma}(\xi) = a(|\xi|)\xi, \tag{1.2}$$

where $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^d$ and where $a : \mathbb{R}_+ \to \mathbb{R}$ is a given function. The function a is assumed below to satisfy a growth condition of the form $a(x) \sim x^{p-2}$ as $x \to +\infty$ for some real number $p \in (1, +\infty)$, so that the natural energy space V for the above model problem is the Sobolev space $W_0^{1,p}(\Omega)$. The data f is taken in $L^q(\Omega)$ where $q := \frac{p}{p-1}$ so that $\frac{1}{p} + \frac{1}{q} = 1$. Hence, the model problem in weak form amounts to finding $u \in V$ such that

$$(\boldsymbol{\sigma}(\nabla u), \nabla v) = (f, v) \quad \forall v \in V, \tag{1.3}$$

where $(\cdot,\cdot)$ denotes the integral over Ω of the (scalar) product of the two arguments. The function a satisfies monotonicity and continuity conditions stated in Section 2 and ensuring that the problem (1.3) is well-posed.

The prototypical example for the present model problem is the so-called p-Laplacian, for which $a(x) = x^{p-2}$. The a priori error analysis for approximating the p-Laplacian by piecewise affine, continuous finite elements has been started by Glowinski and Marrocco [23,24]; see also Ciarlet [15, p. 312]. One well-known difficulty when working with the natural energy norm is that the derived error estimates are not sharp. This drawback has been circumvented by Barrett and Liu [6] upon introducing a so-called quasi-norm, thereby achieving optimal approximation results. The quasi-norm of the error between the exact solution u and the approximate solution, say u_h, is a weighted L^2-norm of the gradient $\nabla(u - u_h)$, where the weight depends on ∇u and ∇u_h.

The a posteriori error analysis of finite element approximations to a large class of nonlinear problems, including the present model problems, has been started by Verfürth; see [33] and [34, p. 47]. The main result is a two-sided bound of the energy error by the dual norm of the residual multiplied by suitable norms of the

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linearized operator at the exact solution, under the assumption that this latter operator is invertible and locally Lipschitz-continuous and that the approximate solution is sufficiently close to the exact solution. This yields in particular residual-based estimators in the energy norm. These estimators have been exploited, in particular, by Veeser [32] to prove the convergence of an adaptive finite element method for the p-Laplacian. Alternatively, quasi-norm error estimates for the p-Laplacian have been analyzed by Liu and Yan [28–30], leading to weighted residual-based estimators. Quasi-norm residual-based estimators have been further explored by Carstensen and Klose [9] with a focus on evaluating the constants in the estimates and under the assumption that the gradient norm of the approximate solution is positive everywhere in the domain. Moreover, gradient recovery techniques have been analyzed by Carstensen et al. [10] to estimate the quasi-norm of the error. Quite recently, Diening and Kreuzer [19] have obtained two-sided bounds for an appropriate measure of the error and proven the linear convergence of a suitable adaptive finite element method. The error measure is the L^2-norm of the difference $\mathbf{F}(\nabla u) - \mathbf{F}(\nabla u_h)$, where the auxiliary vector field $\mathbf{F}$ is such that $\mathbf{F}(\xi) = |\boldsymbol{\sigma}(\xi)|^{\frac{1}{2}}|\xi|^{-\frac{1}{2}}\xi$. This error measure turns out to be equivalent to the quasi-norm of the error, with constants depending on the nonlinearity (that is, the properties of the function a in (1.2)).

We observe that, whatever the error measure, the above bounds on the error involve constants depending on the function a. In the case of the p-Laplacian, they depend on the Lebesgue exponent p. Moreover, with a few exceptions, e.g., [9], the error upper bounds involve unknown generic constants. Therefore, the first objective of this work is to derive *guaranteed* bounds on the error, that is, error upper bounds without undetermined constants, and at the same time ensure *robustness*, that is, two-sided error bounds whose ratio is independent of the nonlinearity. To this purpose, we use as error measure a residual flux-based dual norm, namely

$$\mathcal{J}_u(u_h) = \sup_{v \in V\setminus\{0\}} \frac{(\boldsymbol{\sigma}(\nabla u) - \boldsymbol{\sigma}(\nabla u_h), \nabla v)}{\|v\|_V}.$$

Working with residual flux-based quantities to measure the error is somewhat natural since fluxes satisfy basic conservation properties that are at the heart of approximation methods, even using continuous finite elements. Furthermore, the idea of using a dual norm is inspired by the work of Verfürth where dual norms have been considered, e.g., in the context of parabolic [38] and convection-dominated stationary convection–diffusion equations [40]. Dual residual norms have also been considered for nonlinear problems in [33], and the present dual norm has been considered in [11,12]. More recently, it has been observed in [46] that residual flux-based error measures are also natural in the context of diffusion problems with heterogeneous coefficients. Furthermore, we remark that although our error upper bounds are fully computable, the actual error measure is not, even if the exact solution is known; we will discuss below how the error measure can be approximated in numerical experiments with synthetic exact solutions so as to compute effectivity indices. Note, however, that in practical computations, the exact solution is never known and hence the error is never computable. We also point out that achieving robust error estimates does not mean necessarily that the error bounds can be extended to the limit cases $p = 1$ or $p = +\infty$, similar to the vanishing-diffusion limit in convection–diffusion equations.

Our a posteriori error estimates are formulated in terms of a $\mathbf{H}(\mathrm{div})$-conforming flux reconstruction. For conforming finite element methods, related earlier work in the linear case includes [1] (here the flux is not explicitly reconstructed) and [7,17,27,31]. In the spirit of Luce and Wohlmuth [31], guaranteed a posteriori estimates of the present type were proposed in [45] for the Laplace equation. They have been shown robust for inhomogeneous and anisotropic diffusion in [46] and for the reaction–diffusion case in [13]. We also refer to [22] for a unified setting encompassing various discretization methods in the context of the heat equation. Recently, Verfürth [41] derived another estimate based on flux reconstruction for singularly perturbed diffusion problems and, similar to [13,22,45,46], proved (see [41, Proposition 2.2]) that this estimate is a lower bound for the classical residual one of [34]. In the nonlinear case, the only work deriving a posteriori estimates based on flux reconstruction we are aware of is [26]. Therein, quasi-linear diffusion problems similar to (1.1a)–(1.1b) are discretized by various nonconforming locally conservative methods.

In the present paper, the a posteriori error analysis based on $\mathbf{H}(\mathrm{div})$-conforming flux reconstruction proceeds as follows. The error upper bound hinges on a local conservation property of the reconstructed flux, say $\mathbf{t}_h$; see Assumption 3.4. The error lower bound hinges instead on an approximation property of $\mathbf{t}_h$; see Assumption 4.1. This approximation property enables us to prove that our estimates are lower bounds for the classical residual ones. We provide two examples for reconstructing the flux $\mathbf{t}_h$ satisfying Assumptions 3.4 and 4.1 in the context of piecewise affine, continuous finite elements. Higher-order methods are not considered herein. This is motivated, in part, by the fact that in many cases the exact solution u may not have much additional regularity beyond that of the natural energy space V; see [15, p. 324] for a similar remark concerning the p-Laplacian.

The discrete problem amounts to a system of nonlinear equations, and, in practice, is solved using an iterative method involving some kind of linearization. Given an approximate solution, say $u_{L,h}$, at a given stage of the iterative process and on a given mesh, there are actually two sources of error, namely linearization and discretization. Balancing these two sources of error can be of paramount importance in practice, since it can avoid performing an excessive number of nonlinear solver iterations if the discretization error dominates. Therefore, the second objective of this work is to design a posteriori error estimates distinguishing linearization and discretization errors in the context of an adaptive procedure. This type of analysis has been started by Chaillou and Suri [11,12] for a certain class of nonlinear problems similar to the present one and in the context of iterative solution of linear algebraic systems in [25]; we also refer to [16] for adaptive numerical approximation of nonlinear problems in the wavelets context. Chaillou and Suri only considered a fixed stage of the linearization process, while we take here the analysis one step further in the context of an iterative loop. Furthermore, they only considered a specific form for the linearization, namely of fixed point-type, while we allow for a wider choice, including Newton–Raphson methods. We consider an adaptive loop in which at each step, a fixed mesh is considered and the nonlinear solver is iterated until the linearization error estimate is brought below the discretization error estimate; then, the mesh is adaptively refined and the loop is advanced. In this work, we will not tackle the delicate issue of proving the convergence of the above adaptive algorithm. We will also assume that at each iterate of the nonlinear solver, a well-posed problem is obtained. This property is by no means granted in general; it amounts, for the p-Laplacian, to assume, as mentioned before in [9], that the gradient norm of the approximate solution is positive everywhere in the domain. We mention that in our numerical experiments, all the discrete problems were indeed found to be well-posed.

This paper is organized as follows. Section 2 describes the setting for the nonlinear problem together with its discretization and linearization. Section 3 is devoted to the derivation of the guaranteed error upper bounds, while Section 4 is concerned with the efficiency of the estimators. Section 5 presents two possible approaches to reconstruct the flux $\mathbf{t}_h$ in the context of piecewise affine, continuous finite elements. Section 6 contains the numerical results. Finally, Appendix A collects various auxiliary lemmas.