Comput. Methods Appl. Mech. Engrg. 200 (2011) 2434–2446

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

A mixed formulation of the Bingham fluid flow problem: Analysis and numerical solution

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ARTICLE INFO

Article history:
Received 26 October 2010
Received in revised form 28 February 2011
Accepted 5 April 2011
Available online 12 April 2011

Keywords:
Non-Newtonian flow
Bingham fluid
Mixed formulation
Augmented formulation
Regularization

ABSTRACT

In this paper we introduce a mixed formulation of the Bingham fluid flow problem. We consider both the original and a regularized version of the problem, where a parameter ε is introduced, forcing the entire domain to be formally a fluid region. In general, common solvers for the regularized problem experience a performance degradation when the parameter ε gets smaller. The method studied here introduces an auxiliary tensor variable and shows enhanced numerical properties for small values of ε. A good performance is also observed for the non-regularized case. The well posedness for the regularized problem and the equivalence – at the continuous level – between the original (primitive variables) and the mixed formulation are demonstrated. We analyze properties of linearized problems that are relevant for the convergence of numerical solvers. A finite element method for the mixed formulation is discussed. Numerical results confirm the predicted better performances of the mixed formulation when compared to the primitive variables formulation. A comparison to a non-regularized solver based on the augmented Duvaut–Lions–Glowinski formulation of the problem is carried out as well.

1. Introduction

The Bingham plastic is a material that behaves like a rigid medium for stresses $\boldsymbol{\tau}$ not exceeding a certain critical value τ_s (called the *yield stress*) and behaves like an incompressible fluid if the stresses are equal to or exceed τ_s. The viscosity of the fluid depends on the shear rate, thus the Bingham flow represents an example of a non-Newtonian fluid. Bingham fluids occur in many situations of geophysical as well as industrial interest, see [6] for a comprehensive review, and more recently [37] for the applications in hemodynamics.

Let $D\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$ denote the strain rate tensor and let $|D\mathbf{u}| = \sqrt{D\mathbf{u} : D\mathbf{u}}$ be the Frobenius norm of $D\mathbf{u}$. The conservation of momentum in the steady case for an incompressible fluid reads

$$\begin{cases} -\mathbf{div}\boldsymbol{\tau} + \nabla p = \boldsymbol{f} \\ \nabla \cdot \mathbf{u} = 0 \end{cases} \quad \text{in } \Omega, \tag{1.1}$$

where **div** denotes the divergence operator for tensors, $\mathbf{u}$, p are the unknown velocity and pressure. For Bingham fluids the domain Ω is split into two subdomains, the fluid region Ω_f and the rigid (or plug) region Ω_r. The constitutive relation for the stress deviator tensor $\boldsymbol{\tau}$ and the strain rate tensor reads

$$D\mathbf{u} = \begin{cases} 0 & \text{for } |\boldsymbol{\tau}| \leqslant \tau_s \text{ (rigid region)}, \\ \left(1 - \dfrac{\tau_s}{|\boldsymbol{\tau}|}\right)\dfrac{\boldsymbol{\tau}}{2\mu} & \text{for } |\boldsymbol{\tau}| > \tau_s \text{ (fluid region)}. \end{cases} \tag{1.2}$$

where the plastic viscosity $\mu > 0$ and the yield stress $\tau_s \geqslant 0$ are given constants. These equations can be observed as a generalization of the classical Stokes equation having in Ω_f a shear dependent viscosity $\hat{\mu} = 2\mu + \dfrac{\tau_s}{|D\mathbf{u}|}$ that reduces to the Stokes equations with constant viscosity for $\tau_s = 0$. One of the difficult features of the problem is that the two regions are unknown *a priori* and finding them is a part of the problem; also $\hat{\mu}$ becomes singular in the plug zone. A common way to avoid this difficulty is to regularize $\hat{\mu}$. This can be done in different ways, see e.g., [4,32,18]. Here we consider the Bercovier–Engelman regularization [4]: in the definition of $\hat{\mu}$ the norm $|D\mathbf{u}|$ is replaced with $|D\mathbf{u}|_\varepsilon = \sqrt{D\mathbf{u} : D\mathbf{u} + \varepsilon^2}$. Extension of the approach presented hereafter to other forms of regularization can be considered as well. The regularization ensures $\hat{\mu}$ to be nonsingular even in presence of plug regions and the fluid equations can be posed in the entire domain:

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[1] The work of this author was supported in part by the Russian Foundation for Basic Research grants 09-01-00115, 11-01-00971 and by a Visiting Fellowship granted by the Emerson Center for Scientific Computation at Emory University.

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doi:10.1016/j.cma.2011.04.004

$$\begin{cases} -\mathbf{div}\left(2\mu + \dfrac{\tau_s}{|D\mathbf{u}|_\varepsilon}\right) D\mathbf{u} + \nabla p = \boldsymbol{f} & \text{in } \Omega. \\ -\nabla \cdot \mathbf{u} = 0 \end{cases} \tag{1.3}$$

The regularized model can be treated as the model of a quasi-Newtonian fluid flow and its numerical implementation becomes relatively simple within an existing CFD code. A variety of well-established computational techniques, including parameter free iterative algorithms as Newton method and Krylov subspace methods, can be used to treat the regularized equations numerically. However, the regularization prevents finding the 'exact' visco-plastic solution. In particular, finding arrested states and defining plug regions with $\varepsilon > 0$ become non-trivial tasks, see [35] and also [38,39] (the latter papers deal with compressible fluids). Therefore accurate and predictive computations demand using small values of the regularization parameter ε (see e.g., [18,31,12]). Using small values of ε in (1.3) gives rise to several computational issues. For example, the Newton method applied to the regularized problem (1.3) is not robust with respect to ε (see [12] and numerical results in [25,26]). The domain of convergence for the Newton method shrinks as $\varepsilon \to 0$. Indeed, the norm of the matrix of the second derivatives grows like $O(\varepsilon^{-1})$ [12] implying that to ensure convergence the initial guess for the Newton method should belong to an $O(\varepsilon)$-neighborhood of the (unknown) solution. One possibility to overcome the issue is to apply a continuation method in ε. This means that ε is selected dynamically and it gets smaller along the iterations. Another option is to perform a number of more robust Picard iterations and switch to the Newton method when a sufficiently good approximation to the solution is found. In the latter case, however, the required number of Picard iterations still grows as $\varepsilon \to 0$ [25].

Both mathematical and numerical difficulties forced several authors to consider different formulations of the problem (1.1) and (1.2). One approach is based on the variational inequality formulation of Duvaut and Lions [15] and has been proposed by Glowinski and coauthors (see Section 8 of the review paper [13] and refereneces therein). The formulation has been studied mathematically and used to solve the problem numerically with Uzawa-like iterative schemes. The iterations are proven to be convergent upon the introduction of a relaxation parameter (see [12]), however may exhibit a slow convergence rate. Nevertheless, the approach is attractive for solving practical problems when it is necessary to compute the 'true' visco-plastic solution and find the plug region (see e.g., [35,34]). We briefly review this approach in Section 5.

In this paper, we consider a different formulation intended to enhance the numerical properties of the regularized formulation (1.3). We introduce an auxiliary symmetric tensor W such that

$$|D\mathbf{u}|_\varepsilon W - D\mathbf{u} = \mathbf{0}. \tag{1.4}$$

Equations (1.3) in Ω with the auxiliary variable read

$$\begin{cases} -\mathbf{div}(2\mu D\mathbf{u} + \tau_s W) + \nabla p = \boldsymbol{f}, \\ -\nabla \cdot \mathbf{u} = 0. \end{cases} \tag{1.5}$$

System (1.4) and (1.5) represents the mixed formulation we investigate in this paper. We will show that this formulation is efficient for solving the regularized problem. For a given ε the number of nonlinear iterations required for convergence is significantly reduced compared to solving the original problem (1.3) in the primitive variables. While most analysis of (1.4) and (1.5) is carried out in this paper for $\varepsilon > 0$, numerical results show that the method remains efficient even for the case $\varepsilon = 0$. In this case, the approach and the resulting iterative method compares favorably with the Uzawa type algorithm for the augmented Lagrangian saddle-point formulation of Glowinski et al.

The mixed formulation (1.4) and (1.5) is closely related with the approach of Cea and Glowinski [7] (see also Sections 5–7 in [13]). In that approach a symmetric tensor W satisfying $W : Du = |D\mathbf{u}|$ was introduced in the numerical formulation through the relation

$$W = \mathbf{P}(W + rD\mathbf{u}) \quad \forall r \geqslant 0, \tag{1.6}$$

with the projector $\mathbf{P}$ on the convex set of tensor functions $Z \in (L^2(\Omega))^{d\times d}$ satisfying $|Z| < 1$. The projector is defined by $\mathbf{P}(Z)(\mathbf{x}) := Z(\mathbf{x})[\max\{1, |Z(\mathbf{x})|\}]^{-1}$. The equations (1.5) and (1.6) were solved numerically with the Uzawa type method with W serving for the primal iterated variable and r as a relaxation parameter. Further, a special regularization was introduced in [13] to facilitate the application of a variant of the Newton method. While formulation (1.4) and (1.5) is formally equivalent to (1.5) and (1.6) for $\varepsilon = 0$, it leads to a different variational formulation and finite element solutions, coupled iterative algorithms of Picard and (for $\varepsilon > 0$) Newton may be directly applied. Moreover (1.4) and (1.5) is amenable to common regularizations like the one used in this paper.

The remainder of the paper is organized as follows. Necessary notations and preliminaries are given in Section 1.1. In Section 2, we consider the weak formulations of (1.3) and (1.4)–(1.5) and prove some well-posedness results. Some linearized problems are studied here as well. We prove that the weak formulations of (1.3) and (1.4)–(1.5) are equivalent in the sense that they share the unique solution. At the same time, equivalence does not necessarily hold for the corresponding numerical discretizations. In Section 3 we introduce non-linear iterative methods of Picard and Newton for solving (1.3) and (1.4)–(1.5). Several convergence estimates for the case $\varepsilon > 0$ are proven which suggest the superior properties of (1.4) and (1.5) in building efficient solvers. A finite element discretization method is considered in Section 4, including the discussion of algebraic properties of resulting discrete systems. In Section 5 we briefly recall another method for numerical treatment of the Bingham problem (1.1) based on variational inequalities and augmentation. Several numerical results are presented in Section 6. These results show that the mixed formulation (1.4) and (1.5) leads to much better convergence rates than the primitive variables formulation (1.3). Moreover, it appears that the Picard method for solving (1.4) and (1.5) is applicable with $\varepsilon = 0$ and demonstrates fast convergence even in this limit case. Thus for $\varepsilon = 0$, we include few results of comparison with the Uzawa method for the augmented saddle-point formulation of Glowinski et al. In this section, we also consider a continuation Newton method based on the mixed formulation. Section 7 contains some closing remarks.

1.1. Notations and preliminaries

In what follows, we use the standard notation for the functional spaces we need: for $1 \leqslant p \leqslant \infty$ and $k > 0$, $L^p(\Omega)$, $H^k(\Omega)$ are standard Lebesgue and Sobolev spaces. Also $L^2_0(\Omega)$ denotes the subspace of $L^2(\Omega)$ of functions with zero mean over Ω, $H^1_0(\Omega)$ is the space of functions in $H^1(\Omega)$ with vanishing trace on $\partial\Omega$. The corresponding spaces for (2D or 3D) vectors are denoted in bold, e.g. $\mathbf{L}^2(\Omega)$, $\mathbf{H}^k(\Omega)$ or $\mathbf{H}^1_0(\Omega)$. The subspace of $\mathbf{H}^1_0(\Omega)$ of *divergence free* vector-functions is denoted by $\mathbf{V}$. We use the notation $\mathscr{H}^k(\Omega)$ for tensors whose components are $H^k(\Omega)$ functions. For symmetric tensors, this particularizes to $\mathscr{L}^p_s(\Omega)$, $\mathscr{H}^k_s(\Omega)$. When there is no possibility of confusion, we omit the indication of the domain Ω. The norm in H^k is denoted by $\|\cdot\|_k$, the scalar product and the norm in L^2 is denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, respectively, the same norm and product notation is used for the vector and tensor counterparts of H^k and L^2.

From the vector identities $2\mathbf{div}D = \Delta + \nabla\nabla\cdot$ and $\nabla\nabla\cdot = \Delta + \nabla\times\nabla\times$ with the help of integration by parts one immediately gets the following Korn type inequalities