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Discussion about parameterization in the asymptotic numerical method: Application to nonlinear elastic shells

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ABSTRACT

The Asymptotic Numerical Method (ANM) is a family of algorithms for path following problems based on the computation of truncated vectorial series with respect to a path parameter "a" [B. Cochelin, N. Damil, M. Potier-Ferry, Méthode Asymptotique Numérique, Hermès-Lavoisier, Paris, 2007]. In this paper, we discuss and compare three concepts of parameterizations of the ANM curves i.e. the definition of the path parameter "a". The first concept is based on the classical arc-length parameterization [E. Riks, Some computational aspects of the stability analysis of nonlinear structures, Computer Methods in Applied Mechanics and Engineering, 47 (1984) 219–259], the second is based on the so-called local parameterization [W. C. Rheinboldt, J. V. Burkadt, A Locally parameterized continuation, Acm Transaction on mathematical Software, 9 (1983) 215–235; R. Seydel, A Tracing Branches, World of Bifurcation, Online Collection and Tutorials of Nonlinear Phenomena (http://www.bifurcation.de), 1999; J. J. Gervais, H. Sadiky, A new steplength control for continuation with the asymptotic numerical method, IAM, J. Numer. Anal. 22, No. 2, (2000) 207–229] and the third is based on a minimization condition of a rest [S. Lopez, An effective parametrization for asymptotic extrapolation, Computer Methods in Applied Mechanics and Engineering, 189 (2000) 297–311]. We demonstrate that the third concept is equivalent to a maximization condition of the ANM step lengths. To illustrate the performance of these proposed parameterizations, we give some numerical comparisons on nonlinear elastic shell problems.

1. Introduction

The asymptotic numerical method (ANM) is a family of algorithms for path following problems [6]. The principle is simply to expand the unknown $(\mathbf{U}, \lambda)$ of a nonlinear problem $\mathbf{R}(\mathbf{U}, \lambda) = 0$ in power series with respect to a path parameter "a":

$$(\mathbf{U}(a), \lambda(a)) = \left(\mathbf{U}^j, \lambda^j\right) + \sum_{i=1}^{N} a^i (\mathbf{U}_i, \lambda_i) \quad , \quad a \in [0, a_{max}] \tag{1}$$

where $(\mathbf{U}^j, \lambda^j)$ is a known and regular solution corresponding to $a = 0$ and N is the truncated order of the series. The interval of validity $[0, a_{max}]$ is deduced from the computation of the truncated vectorial series Eq. (1). So, the step lengths are computed a posteriori by the two following estimations of a_{max} which have been proposed in [4,6]:

$$a_{max} = \left(\epsilon_d \frac{||\mathbf{U}_1||}{||\mathbf{U}_N||}\right)^{\frac{1}{N-i}} \tag{2}$$

$$a_{max} = \left(\epsilon_r \frac{1}{||\mathbf{F}^{nl}_{N+1}||}\right)^{\frac{1}{N+1}} \tag{3}$$

where ϵ_d and ϵ_r are given tolerance parameters, $\mathbf{F}^{nl}_{N+1}$ are the ANM right hand sides [6] (see Eq. (5)) and $||.||$ indicates the standard norm associated with the scalar product. In the first estimation (2), we require that the last term of the truncated series is very small as compared to the first term with a maximal ratio ϵ_d. In the second estimation (3), we require that the norm of an approximation of the residual, $||\mathbf{R}(\mathbf{U}(a), \lambda(a))|| \simeq a^{N+1}||\mathbf{F}^{nl}_{N+1}||$, is lower than a given tolerance ϵ_r. By using the evaluation of the series at $a = a_{max}$, we obtain a new starting point and define, in this way, the ANM continuation procedure. This continuation method has been proved to be an efficient method to compute the solution of nonlinear partial differential equations [1–6,9,10,17,19,20].

The step lengths depend on the definition of the path parameter "a" and we must add an auxiliary equation to define this parameter. The importance of a good choice of the path parameter, in the asymptotic expansions for structural mechanics, has been discussed for the first time in [27]. In the ANM continuation [6], we often used the classical pseudo arc-length parameterization (see Eq. (8)). This parameterization is generally used in the Newton-Raphson methods [7,8,11,15,16,18,21,25,28].

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Another choice, the local parameterization [22,23,25,26] has been examined by Gervais and Sadiky [12,24] in the context of the ANM (see Eq. (10) or (11)).

Lopez [18] proposed a predictor–corrector algorithm. The predictor and the corrector are performing by using asymptotic expansions with small orders. At each order, the parameterization has been based on the minimization of the norm of residual.

In this paper, we discuss about the parameterization in the ANM which can lead to larger step lengths Eq. (2) or (3). Firstly, we introduce some techniques to define local parameterizations. With these local parameterizations, the auxiliary equations are the same as the one used in [12,13,22,23,25,26].

Secondly, we propose the parameterizations based on the terms $\mathbf{U}_p$ or on the right hand sides $\mathbf{F}^{nl}_{p+1}$. At each order p, we compute the unknown $(\mathbf{U}_p, \lambda_p)$ by the linear system (5) and a minimal condition on the norm $\|\mathbf{U}_p\|$ or as in Lopez [18], on the norm $\|\mathbf{F}^{nl}_{p+1}\|$. There is a difference of one order of truncation between these two parameterizations. $\mathbf{U}_p$ is the solution of the problem at order p and $\mathbf{F}^{nl}_{p+1}$ is the right hand side of the problem at order $p+1$. One can demonstrate that, at each order, the parameterizations based on the minimal condition on the norm $\|\mathbf{U}_p\|$ and on the norm $\|\mathbf{F}^{nl}_{p+1}\|$ maximize the ANM step lengths defined, respectively, by Eqs. (2) and (3). A mathematical connection between the parameterization based on the minimal condition on the norm $\|\mathbf{U}_p\|$ and a sort of pseudo arc–length parameterization will be given.

In the Section 2, we remind the basis of ANM. In the Section 3, we introduce some methods to define the auxiliary equation corresponding to local parameterization. In the Section 4, we propose new parameterizations based on a minimal condition of the rest $\mathbf{U}_p$ or of the rest $\mathbf{F}^{nl}_{p+1}$. In the Section 5, these strategies of the choice of the parameterizations are applied on some examples from nonlinear elastic shells.

2. Parameterization with respect to arc–length parameter

Let us consider the following class of nonlinear quadratic problems:

$$\mathbf{R}(\mathbf{U},\lambda) = \mathbf{L}(\mathbf{U}) + \mathbf{Q}(\mathbf{U},\mathbf{U}) - \lambda\mathbf{F} = 0 \qquad (4)$$

where $\mathbf{L}(.)$ and $\mathbf{Q}(.,.)$ are linear and quadratic operators, $\mathbf{F}$ is a given vector and $\mathbf{R}$ is the so-called residual vector. In the case of thin elastic shell equilibrium equations, the unknowns are $\mathbf{U}=(\mathbf{u},\mathbf{S})$ and λ; $\mathbf{u}$ is the displacement and $\mathbf{S}$ is the second Piola-Kirchhoff stress tensor and λ is the load parameter. In this paper, we limit our study to the quadratic framework Eq. (4). More difficult problems can be found in [1,2,6,20]. In these papers, some ideas have been proposed to transform strongly nonlinear problems into quadratic ones. The ANM has been proved to be an efficient method to compute solution path of Eq. (4). The first step is the expansion of the unknown $(\mathbf{U},\lambda)$ with respect to a path parameter "a" as in Eq. (1). In this way, the Eq. (1) defines the solution branch^j (one step of the ANM). By introducing Eq. (1) into Eq. (4) and equating like powers of "a", we obtain the following set of linear problems:

$$\mathbf{L}_t\left(\mathbf{U}_p\right) - \lambda_p\mathbf{F} = -\sum_{r=1}^{p-1}\mathbf{Q}\left(\mathbf{U}_r,\mathbf{U}_{p-r}\right) = \mathbf{F}^{nl}_p \qquad (5)$$

in the unknowns $(\mathbf{U}_p, \lambda_p)$, where $\mathbf{L}_t(.) = \mathbf{L}(.) + 2\mathbf{Q}(\mathbf{U}^j, .)$ is the tangent operator. For $p=1$, we have $\mathbf{F}^{nl}_1 = 0$ and Eq. (5) defines the tangent vector $(\mathbf{U}_1, \lambda_1)$ at $(\mathbf{U}^j, \lambda^j)$ point. The next step is the computation of the terms $(\mathbf{U}_p, \lambda_p)$. The problems Eq. (5) are discretized by a classical finite element method. Let us note that these problems have the same tangent stiffness matrix and hence the terms $(\mathbf{U}_p, \lambda_p)$ $(1 \le p \le N)$ of Eq. (1) are computed by inverting only one stiffness matrix. The last step is the continuation technique [4,6]. The end of the branch^j is the starting point of the next branch^{j+1}.

The general solution of Eq. (5) can be written:

$$\mathbf{U}_p = \lambda_p\hat{\mathbf{U}}_1 + \mathbf{U}^{nl}_p \qquad (6)$$

with

$$\hat{\mathbf{U}}_1 = \mathbf{L}_t^{-1}\mathbf{F} \quad \text{and} \quad \mathbf{U}^{nl}_p = \mathbf{L}_t^{-1}\mathbf{F}^{nl}_p. \qquad (7)$$

To solve system (5) for $p \ge 1$, we must add an auxiliary equation to define the path parameter "a". Various choices are possible; the most used in the ANM algorithms [6] is a Riks pseudo arc–length parameterization (PAL) [21]:

$$a = \langle\mathbf{V},\mathbf{T}\rangle = \left(\mathbf{u}-\mathbf{u}^j\right).\mathbf{u}_1 + \alpha\left(\lambda-\lambda^j\right)\lambda_1 \qquad (8)$$

where $\langle .,.\rangle$ is a scalar product, $\mathbf{T} = {}^t(\mathbf{u}_1, \lambda_1)$ is the tangent vector at the starting point $(\mathbf{U}^j, \lambda^j)$, $\mathbf{V} = (\mathbf{u}-\mathbf{u}^j, \lambda-\lambda^j)$ and $\alpha=1$ or $\alpha=0$. The system (Eqs. (5) and (8)) is solved by Eq. (6) and

$$\lambda_1 = \pm\frac{1}{\sqrt{\alpha + \hat{\mathbf{u}}_1.\hat{\mathbf{u}}_1}}, \lambda_p = -\frac{\hat{\mathbf{u}}_1.\hat{\mathbf{u}}^{nl}_p}{\alpha + \hat{\mathbf{u}}_1.\hat{\mathbf{u}}_1} = -\lambda_1\mathbf{u}_1.\mathbf{u}^{nl}_p. \qquad (9)$$

The sign $\pm$ depends on the choice of the orientation. In this end, we choose the same sign as the one of the scalar product $\langle\mathbf{T}^*(j), \mathbf{T}^*(j-1)\rangle$, where $\mathbf{T}^*(j)$ and $\mathbf{T}^*(j-1)$ are the tangent vector normalized at the starting point of the branch^j and the branch^{j-1}, respectively.

Other choices of the auxiliary equation are discussed in Section 3.

3. Local parameterizations

Another way to parameterize the branch is to use any component of the vector $\mathbf{V}$ as a parameter [26]:

$$a = \langle\mathbf{V}, \mathbf{e}_i\rangle \qquad (10)$$

where $\mathbf{e}_i$ is the ith vector of the canonical basis of $\Re^{n+1}$, n is the degree of freedom of the discretized structure, $(i=1, ..., n+1)$. The case $i=n+1$ corresponds to a load parameter, the others correspond to a displacement parameter. The index "i" and hence the parameterization Eq. (10) has a local nature; i.e. the component "i" should be valid for the branch^j and may be changed for the next branch^{j+1}. With this local parameterization, the auxiliary Eq. (8) is replaced by Eq. (10). In the local parameterization used in [12,22], the auxiliary Eq. (8) is replaced by:

$$a\langle\mathbf{T}, \mathbf{e}_i\rangle = \langle\mathbf{V}, \mathbf{e}_i\rangle \qquad (11)$$

where $\mathbf{T}^* = \pm\frac{1}{\sqrt{1+\hat{\mathbf{u}}_1\cdot\hat{\mathbf{u}}_1}}\begin{Bmatrix}\hat{\mathbf{u}}_1\\ 1\end{Bmatrix}$ is the tangent vector normalized at the point $(\mathbf{u}^j, \lambda^j)$. The vector of projection $\mathbf{e}_i$ must verify the relation $\langle\mathbf{T}^*, \mathbf{e}_i\rangle \neq 0$.

Hence, the system (Eqs. (5) and (10)) is solved by Eq. (6) and

$$\lambda_1 = \frac{1}{\langle\hat{\mathbf{V}}_1, \mathbf{e}_i\rangle} = , \lambda_p = -\lambda_1\frac{\langle\mathbf{v}^{nl}_p, \mathbf{e}_i\rangle}{\langle\mathbf{T}, \mathbf{e}_i\rangle} \qquad (12)$$

with

$$\hat{\mathbf{v}}_1 = {}^t(\hat{\mathbf{u}}_1, 1) \quad \text{and} \quad \mathbf{v}^{nl}_p = {}^t\left(\mathbf{u}^{nl}_p, 0\right) \qquad (13)$$

and the system (Eqs. (5) and (11)) is solved by Eq. (6) and

$$\lambda_1 = \frac{\langle\mathbf{T}^*, \mathbf{e}_i\rangle}{\langle\hat{\mathbf{V}}_1, \mathbf{e}_i\rangle} = \pm\frac{1}{\sqrt{1+\hat{\mathbf{u}}_1.\hat{\mathbf{u}}_1}}, \lambda_p = -\lambda_1\frac{\langle\mathbf{v}^{nl}_p, \mathbf{e}_i\rangle}{\langle\mathbf{T}, \mathbf{e}_i\rangle}. \qquad (14)$$