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Efficient mesh optimization schemes based on Optimal Delaunay Triangulations ☆

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ABSTRACT

In this paper, several mesh optimization schemes based on Optimal Delaunay Triangulations are developed. High-quality meshes are obtained by minimizing the interpolation error in the weighted L^1 norm. Our schemes are divided into classes of local and global schemes. For local schemes, several old and new schemes, known as mesh smoothing, are derived from our approach. For global schemes, a graph Laplacian is used in a modified Newton iteration to speed up the local approach. Our work provides a mathematical foundation for a number of mesh smoothing schemes often used in practice, and leads to a new global mesh optimization scheme. Numerical experiments indicate that our methods can produce well-shaped triangulations in a robust and efficient way.

1. Introduction

We shall develop fast and efficient mesh optimization schemes based on Optimal Delaunay Triangulations (ODTs) [1–3]. Let ρ be a given density function defined on a convex domain $\Omega \subset \mathbb{R}^n$, i.e. $\rho > 0, \int_\Omega \rho dx < \infty$. Let $\mathcal{T}$ be a simplicial triangulation of Ω, and let $u(\boldsymbol{x}) = \|\boldsymbol{x}\|^2$ and u_I the piecewise linear nodal interpolation of u based on $\mathcal{T}$. We associate the following weighted L^1 norm of the interpolation error as an energy to the mesh $\mathcal{T}$

$$E(\mathcal{T}) = \int_\Omega |(u_I - u)(\boldsymbol{x})|\rho(\boldsymbol{x})d\mathbf{x}.$$

Let $\mathcal{T}_N$ denote the set of all triangulations with at most N vertices. Our mesh optimization schemes will be derived as iterative methods for solving the following optimization problem:

$$\inf_{\mathcal{T}\in\mathcal{T}_N} E(\mathcal{T}). \tag{1.1}$$

Minimizers of (1.1) will be called Optimal Delaunay Triangulations.

Mesh optimization by minimizing some energy, also known as the variational meshing method, has been studied by many authors; see, e.g. [4–7] and references therein. There are many energies proposed in the literature for this purpose, including the widely used harmonic energy in moving mesh methods [8–10], summation of weighted edge lengths [11,12], and the distortion energy used in the approach of Centroid Voronoi Tessellation (CVT) [13,14]. The advantages of our approach are:

1. Mathematical analysis is provided to show minimizers of (1.1) will try to equidistribute the mesh size according to the density function as well as preserve the shape regularity.
2. Optimization of the connectivity of vertices is naturally included in our optimization problem.
3. Efficient algorithms, including local and global mesh optimization schemes, are developed for the optimization problem (1.1).

To solve the optimization problem (1.1), we decompose it into two sub-problems. Let us denote a triangulation $\mathcal{T}_N$ by a pair (p,t), where $p \in \Omega^N$ represents the set of N vertices and t represents the connectivity of vertices, i.e. how vertices are connected to form simplexes, and rewrite the energy as $E(p,t)$. We solve the following two sub-problems iteratively:

1. `Fix the location of vertices and solve` $\min_t E(p,t)$;
2. `Fix the connectivity of vertices and solve` $\min_p E(p,t)$.

We stress from the outset that both problems $\min_t E(p,t)$ and $\min_p E(p,t)$ do not need to be solved exactly. We are not interested in the *optimal* mesh but rather meshes with good geometric quality (including the density of vertices and shape regularity of simplices). We shall show that the mesh quality will be considerably improved by performing just a few steps of the iteration methods developed in this paper.

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Let us first consider the optimization problem $\min_t E(p,t)$. That is, for a fixed vertex set p, find the optimal connectivity of the vertices (in the sense of minimizing the weighted interpolation error $E(\mathcal{T})$). In [2,3], we proved that when Ω is convex, the minimizer is a Delaunay triangulation of the point set p. Thus, the problem $\min_t E(p,t)$ is simplified to:

$$\text{Given a set of vertices } p, \text{construct a Delaunay triangulation of } p. \quad (1.2)$$

The problem (1.2) is well studied in the literature [15,16]. We can classify methods proposed for (1.2) as one of

- Local method: edge or face flipping;
- Global method: lifting method (QHULL).

The focus of this paper is on the optimization problem $\min_p E(p,t)$, namely optimizing the placement of vertices when the connectivity is fixed. We shall also discuss two types of methods:

- Local mesh smoothing;
- Global mesh optimization.

Local relaxation methods are commonly used methods for mesh improvement. For example, Gauss–Seidel-type relaxation methods consider a local optimization problem by moving only one vertex at a time. The vertex is moved inside the domain bounded by its surrounding simplexes while keeping the same connectivity to improve geometric mesh quality such as angles or aspect ratios. This is known as mesh smoothing in the meshing community [1,17–23,13]. With several formulas for the interpolation error, we shall derive a suitable set of local mesh smoothing schemes among which the most popular scheme, Laplacian smoothing, will be derived as a special case.

Local methods, however, can only capture the high frequency in the associated energy, and thus results in slow convergence when the number of grid points becomes larger; see [12,24] for related discussions and numerical examples. To overcome slow convergence of local mesh smoothing schemes, some sophisticated multigrid-like methods, notably full approximation scheme (FAS), have been recently proposed [25–29]. To use multigrid-type methods, one has to generate and maintain a nested mesh hierarchy which leads to complex implementations with large memory requirements. The interpolation of point locations from the coarse grid to the fine grid can fold triangulations, and addressing this carefully leads to additional implementation complexity. See [27,25] for related discussions.

We shall derive a global mesh optimization method by using another technique of multilevel methods: multilevel preconditioners. One iteration step of our method reads as

$$p^{k+1} = p^k - A^{-1}\nabla E(p^k, t), \quad (1.3)$$

where A is a graph Laplacian matrix with nice properties: it is symmetric and positive definite (SPD) and also a diagonally dominant M-matrix. Note that if we replace A by $\nabla^2 E(p^n)$ in (1.3), it becomes Newton's method. Our choice of A can be thought as a preconditioner of the Hessian matrix. Comparing with Newton's method, our choice of A has several advantages

- A is easy to compute, while $\nabla^2 E$ is relatively complicated;
- A^{-1} can be computed efficiently using algebraic multigrid methods (AMG) since A is an SPD and M-matrix, while $\nabla^2 E$ may not be;
- A is a good approximation of $\nabla^2 E$.

We should clarify that our methods are designed for mesh optimization, not mesh generation. Therefore, we only move interior nodes and assume all boundary nodes are well placed to capture the geometry of the domain. We note that many mesh generators become slow when the number of vertices becomes large. Therefore, we call mesh generators only to generate a very coarse mesh, and then apply our mesh optimization methods to the subsequently refined meshes. By doing so, we can generate high-quality meshes with large numbers of elements in an efficient way.

The concept of Optimal Delaunay Triangulation (ODT) was introduced in [2] and some local mesh smoothing schemes were reported in a conference paper [1] and summarized in the first author's Ph. D thesis [3]. Application of ODT to other problems can be found in [30–32]. In this paper, we include some results from [1,3] for the completeness and more importantly, present several new improvements listed below:

- several improved smoothing schemes for non-uniform density functions;
- a neat remedy for possible degeneration of elements near the boundary;
- a global mesh optimization scheme;
- some 3D numerical examples.

The rest of this paper is organized as follows. In Section 2, we review the theory on Delaunay and Optimal Delaunay Triangulations. In Section 3, we go over algorithms for the construction of Delaunay triangulation. In Section 4, we give several formulae on the energy and its derivatives. Based on these formulae, we present several optimization schemes including local mesh smoothing and a global modified Newton method. In Section 5, we provide numerical examples to show the efficiency of our methods. In the last section, we conclude and discuss future work.

2. Delaunay and Optimal Delaunay Triangulations

Delaunay triangulation (DT) is the most commonly used unstructured triangulation in many applications. It is often defined as the dual of the Voronoi diagram [33]. In this section we use an equivalent definition [34,35] which involves only the triangulation itself.

Let V be a finite set of points in $\mathbb{R}^n$. The convex hull of V, denoted by $CH(V)$, is the smallest convex set which contains these points.

Definition 2.1. A Delaunay triangulation of V is a triangulation of $CH(V)$ so that it satisfies empty sphere condition: there are no points of V inside the circumsphere of any simplex in the triangulation.

There are many characterizations of Delaunay triangulations. In two dimensions. Sibson [36] observed that Delaunay triangulations maximize the minimum angle of any triangle. Lambert [37] showed that Delaunay triangulations maximize the arithmetic mean of the radius of inscribed circles of the triangles. Rippa [38] showed that Delaunay triangulations minimize the Dirichlet energy, i.e. the integral of the squared gradients. D'Azevedo and Simpson [39] showed that in two dimensions, Delaunay triangulations minimizes the maximum containing radius (the radius of the smallest sphere containing the simplex). Rajan [40] generalized this characterization to higher dimensions. Chen and Xu [2] characterize Delaunay triangulations from a function approximation point of view. We shall briefly survey the approach by Chen and Xu [2] in the following.

Definition 2.2. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\mathcal{T}$ a triangulation of Ω, and $u_{I,\mathcal{T}}$ be the piecewise linear and globally continuous nodal interpolation of a given function $u \in \mathcal{C}(\bar{\Omega})$ based on the triangulation $\mathcal{T}$. Let $1 \leqslant q \leqslant \infty$. We define an error-based mesh quality $Q(\mathcal{T}, u, q)$ as