Comput. Methods Appl. Mech. Engrg. 200 (2011) 101–113

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

A posteriori error analysis of an augmented mixed formulation in linear elasticity with mixed and Dirichlet boundary conditions

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ARTICLE INFO

Article history:
Received 30 October 2009
Received in revised form 30 July 2010
Accepted 30 July 2010
Available online 27 August 2010

MSC:
65N15
65N30
65N50
74B05
74S05

Keywords:
Mixed finite element
Augmented formulation
A posteriori error estimator
Linear elasticity

ABSTRACT

We develop a residual-based a posteriori error analysis for the augmented mixed methods introduced in [13,14] for the problem of linear elasticity in the plane. We prove that the proposed a posteriori error estimators are both reliable and efficient. Numerical experiments confirm these theoretical properties and illustrate the ability of the corresponding adaptive algorithms to localize the singularities and large stress regions of the solutions.

1. Introduction

Recently, a new stabilized mixed finite element method was presented and analyzed in [13] for the problem of linear elasticity in the plane assuming pure homogeneous Dirichlet boundary conditions and mixed boundary conditions with non-homogeneous Neumann data. This approach was extended to the case of pure non-homogeneous Dirichlet boundary conditions in the subsequent work [14].

The augmented formulations proposed in [13,14] rely on the mixed method of Hellinger and Reissner that provides simultaneous approximations of the displacement $\mathbf{u}$ and the stress tensor $\boldsymbol{\sigma}$. The symmetry of $\boldsymbol{\sigma}$ is imposed weakly through the use of a Lagrange multiplier, which enters the system as a new variable that can be interpreted as the rotation $\gamma := \frac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^t)$ (see [1,19]). When mixed boundary conditions are considered, the essential one (Neumann) is also imposed weakly, which yields the introduction of the trace of the displacement on the Neumann boundary as a Lagrange multiplier (see [5]).

Although the usual dual-mixed variational formulations satisfy the hypotheses of the Babuška–Brezzi theory, it is difficult to derive explicit finite element subspaces yielding stable discrete schemes. In particular, when mixed boundary conditions with non-homogeneous Neumann data are imposed, the PEERS elements can be applied but they yield a non-conforming Galerkin scheme. This was one of the main motivations to introduce the augmented formulation from [13].

The approach there is based on the introduction of suitable Galerkin least-squares terms that arise from the constitutive and equilibrium equations, and from the relation defining the rotation in terms of the displacement. In [14], besides these Galerkin least-squares terms, a consistency term related with the non-homogeneous Dirichlet boundary condition is added. In the case of pure Dirichlet boundary conditions, the bilinear form of the augmented formulation is bounded and coercive on the whole space and hence, the associated Galerkin scheme is well-posed for *any* finite element subspace. Thus, it is possible to use as finite element subspaces some non-feasible choices for the usual (non-augmented) dual-mixed formulation. In particular, it is possible to employ Raviart–Thomas elements of lowest order to approximate the stress tensor, continuous piecewise linear elements for the displacement, and piecewise

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doi:10.1016/j.cma.2010.07.016

constants for the rotation. In the case of mixed boundary conditions, the trace of the displacement on the Neumann boundary can be approximated by continuous piecewise linear elements on an independent partition of that boundary whose mesh size needs to satisfy a compatibility condition with the mesh size of the triangulation of the domain.

As pointed out in [13,14], when uniform triangulations are used the mixed finite element schemes proposed there are cheaper than the classical PEERS and BDM elements. More precisely, in the lowest order case the total number of unknowns (dof) for the augmented scheme behaves asymptotically as $5\bar{m}$, where $\bar{m}$ is the number of triangles in the triangulation; for PEERS and BDM (with a static condensation process), the total number of dof behaves asymptotically as $7.5\bar{m}$ and $9\bar{m}$, respectively (see Section 5 in [7] for more details). On the other hand, the lowest order symmetric mixed finite element proposed recently in [4] consists of piecewise linear displacements and piecewise quadratic stresses augmented with some cubic functions and involves 30 dof per triangle; the total number of unknowns in this case behaves asymptotically as $11.5\bar{m}$. More recently, a mixed finite element method with weakly imposed symmetry has been proposed in [2]. In the lowest order case, the stresses are approximated by the Cartesian product of two copies of the BDM finite element space and the displacements and rotations are approximated by piecewise constants; the total number of dof in this case behaves asymptotically as $9\bar{m}$. A reduced element involving asymptotically $7.5\bar{m}$ dof was also presented in [2]. We must also mention that the approach in [13,14] was recently extended in [16] to the 3D linear elasticity problem with pure Dirichlet boundary conditions. This approach seems to be advantageous as compared with the mixed finite element method from [3] (see [16] for more details).

Motivated by the competitive character of the augmented scheme introduced in [13], an a posteriori error analysis of residual type was developed in [7] in the case of pure homogeneous Dirichlet boundary conditions. In this paper, we extend the analysis in [7] to the augmented schemes introduced in [13] for the case of mixed boundary conditions and in [14] for non-homogeneous Dirichlet boundary conditions.

The rest of the paper is organized as follows: In Section 2, we recall the continuous and discrete augmented formulations proposed in [13] for problem (2.1). We develop a residual-based a posteriori error analysis and show that the a posteriori error estimator is both reliable and efficient. Then, in Section 3, we recall from [14] the augmented variational and discrete schemes proposed in the case of non-homogeneous Dirichlet boundary conditions, and deduce an a posteriori error estimator of residual type which is shown to be both reliable and efficient. Finally, in Section 4 we provide several numerical results that illustrate the performance of the augmented Galerkin schemes and confirm the theoretical properties of the a posteriori error estimators introduced in this paper. Moreover, numerical experiments show that the adaptive algorithms based on these a posteriori error estimators are able to localize the singularities and large stress regions of the solutions.

Notation and preliminary results. Given any Hilbert space H, we denote by H^2 and $H^{2\times 2}$, respectively, the spaces of vectors and square tensors of order 2 with entries in H. In particular, given $\boldsymbol{\tau} := (\tau_{ij})$ and $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we denote $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})$, $\mathrm{tr}(\boldsymbol{\tau}) := \tau_{11} + \tau_{22}$ and $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\zeta_{ij}$. In addition, given differentiable scalar, vector and tensor fields, ϕ, $\mathbf{v} = (v_i) \in \mathbb{R}^2$ and $\boldsymbol{\tau} := (\tau_{ij}) \in \mathbb{R}^{2\times 2}$,

$$\mathbf{curl}(\phi) := \begin{pmatrix} -\frac{\partial\phi}{\partial x_2} \\ \frac{\partial\phi}{\partial x_1} \end{pmatrix}, \quad \underline{\mathbf{curl}}(\mathbf{v}) := \begin{pmatrix} \mathbf{curl}(v_1)^{\mathtt{t}} \\ \mathbf{curl}(v_2)^{\mathtt{t}} \end{pmatrix},$$

$$\mathbf{curl}(\boldsymbol{\tau}) := \begin{pmatrix} \frac{\partial\tau_{12}}{\partial x_1} - \frac{\partial\tau_{11}}{\partial x_2} \\ \frac{\partial\tau_{22}}{\partial x_1} - \frac{\partial\tau_{21}}{\partial x_2} \end{pmatrix}.$$

Let $\Omega \subset \mathbb{R}^2$ be a bounded and simply connected domain with polygonal boundary Γ, and let Γ_D and Γ_N be two disjoint subsets of Γ such that Γ_D has positive measure and $\Gamma = \overline{\Gamma}_D \cup \overline{\Gamma}_N$. We use the standard terminology for Sobolev spaces and norms. We denote $H^1_{\Gamma_D}(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$, $H(\mathbf{div};\Omega) := \{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \mathbf{div}(\boldsymbol{\tau}) \in [L^2(\Omega)]^2\}$ and $[L^2(\Omega)]^{2\times 2}_{\mathtt{skew}} := \{\boldsymbol{\eta} \in [L^2(\Omega)]^{2\times 2} : \boldsymbol{\eta} + \boldsymbol{\eta}^{\mathtt{t}} = \mathbf{0}\}$. We recall that $[H^{-1/2}(\Gamma_N)]^2$ is the dual of the space $[H^{1/2}_{00}(\Gamma_N)]^2 := \{\mathbf{v}|_{\Gamma_N} : \mathbf{v} \in [H^1(\Omega)]^2, \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\}$ and denote by $\langle\cdot,\cdot\rangle_{\Gamma_N}$ the associated duality pairing with respect to the $[L^2(\Gamma_N)]^2$-inner product; cf. [18].

Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$. We assume that for all $h > 0$, $\overline{\Omega} = \cup\{T : T \in \mathcal{T}_h\}$ and each point in $\overline{\Gamma}_D \cap \overline{\Gamma}_N$ is a vertex of $\mathcal{T}_h$. Given a triangle $T \in \mathcal{T}_h$, we denote by h_T its diameter and define the mesh size $h := \max\{h_T : T \in \mathcal{T}_h\}$; we denote by $E(T)$ the set of the edges of T, and by E_h the set of all the edges of triangles in the triangulation $\mathcal{T}_h$. Then, we can write $E_h = E_h(\Omega) \cup E_h(\Gamma_D) \cup E_h(\Gamma_N)$, where $E_h(S) := \{e \in E_h : e \subseteq S\}$ for $S \subset \mathbb{R}^2$. Given an edge $e \in E_h$, we denote by h_e the length of e. In addition, given an integer $\ell \geqslant 0$ and a subset S of $\mathbb{R}^2$, we denote by $\mathcal{P}_\ell(S)$ the space of polynomials in two variables defined in S of total degree at most ℓ, and for each $T \in \mathcal{T}_h$, we define the local Raviart–Thomas space of order zero $\mathcal{RT}_0(T) := \mathrm{span}\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{x}\} \subseteq [\mathcal{P}_1(T)]^2$, where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the canonical basis of $\mathbb{R}^2$ and $\mathbf{x}$ is a generic vector of $\mathbb{R}^2$. Finally, we use C or c, with or without subscripts, to denote generic constants, independent of the discretization parameters, which may take different values at different occurrences.

In order to prove the reliability of the a posteriori error estimators, we will make use of the well-known Clément interpolation operator, $I_h : H^1(\Omega) \to X_h$ (see [12]), where X_h is the space of continuous, piecewise linear functions on $\mathcal{T}_h$. We recall that I_h is defined so that $I_h(v) \in X_h \cap H^1_{\Gamma_D}(\Omega)$ for all $v \in H^1_{\Gamma_D}(\Omega)$. The standard local approximation properties stated in the following lemma are proved in [12].

Lemma 1.1. *There exist positive constants c_1, c_2, independent of h, such that for all $\varphi \in H^1(\Omega)$ there hold*

$$\|\varphi - I_h(\varphi)\|_{L^2(T)} \leqslant c_1 h_T \|\varphi\|_{H^1(\Delta(T))}, \quad \forall T \in \mathcal{T}_h,$$

$$\|\varphi - I_h(\varphi)\|_{L^2(e)} \leqslant c_2 h_e^{1/2} \|\varphi\|_{H^1(\Delta(e))}, \quad \forall e \in E_h,$$

where $\Delta(T) := \cup\{T' \in \mathcal{T}_h : T' \cap T \neq \emptyset\}$ *and* $\Delta(e) := \cup\{T' \in \mathcal{T}_h : T' \cap e \neq \emptyset\}$.

To prove the efficiency of the a posteriori error estimators, we proceed as in [9,10], and use inverse inequalities and the localization technique introduced in [21], which is based on triangle-bubble and edge-bubble functions. Given $T \in \mathcal{T}_h$ and $e \in E(T)$, we let ψ_T and ψ_e be the usual triangle-bubble and edge-bubble functions (see (1.5) and (1.6) in [21], respectively). In particular, $\psi_T \in \mathcal{P}_3(T)$, $\mathrm{supp}(\psi_T) \subset T$, $\psi_T = 0$ on ∂T, and $0 \leqslant \psi_T \leqslant 1$ in T. Similarly, $\psi_e|_T \in \mathcal{P}_2(T)$, $\mathrm{supp}(\psi_e) \subseteq \omega_e := \cup\{T' \in \mathcal{T}_h : e \in E(T')\}$, $\psi_e = 0$ on $\partial T \backslash e$, and $0 \leqslant \psi_e \leqslant 1$ in ω_e. We also recall from [20] that, given $k \in \mathbb{N}$, there exists an extension operator $L : C(e) \to C(T)$ such that for all $p \in \mathcal{P}_k(e)$, $L(p) \in \mathcal{P}_k(T)$ and $L(p)|_e = p$. In the following lemma we collect some additional properties of ψ_T, ψ_e and L.

Lemma 1.2. *Let $k \in \mathbb{N}$. For any triangle T, there exist positive constants c_1, c_2, c_3 and c_4, depending only on k and the shape of T, such that for all $q \in \mathcal{P}_k(T)$ and $p \in \mathcal{P}_k(e)$, there hold*

$$\|\psi_T q\|_{L^2(T)} \leqslant \|q\|_{L^2(T)} \leqslant c_1 \|\psi_T^{1/2} q\|_{L^2(T)}, \tag{1.1}$$

$$\|\psi_e p\|_{L^2(e)} \leqslant \|p\|_{L^2(e)} \leqslant c_2 \|\psi_e^{1/2} p\|_{L^2(e)}, \tag{1.2}$$

$$c_4 h_e^{1/2} \|p\|_{L^2(e)} \leqslant \|\psi_e^{1/2} L(p)\|_{L^2(T)} \leqslant c_3 h_e^{1/2} \|p\|_{L^2(e)}. \tag{1.3}$$

Proof. See Lemma 4.1 in [20]. □