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Analysis of a variational multiscale method for Large–Eddy simulation and its application to homogeneous isotropic turbulence

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ABSTRACT

In this paper a variational multiscale method based on local projection and grad–div stabilization for Large–Eddy simulation for the incompressible Navier–Stokes problem is considered. An a priori error estimate is given for a case with rather general nonlinear (piecewise constant) coefficients of the subgrid models for the unresolved scales of velocity and pressure. Then the design of the subgrid scale models is specified for the case of homogeneous isotropic turbulence and studied for the standard benchmark problem of decaying homogeneous isotropic turbulence.

1. Introduction

Incompressible viscous flows of a Newtonian fluid are modeled by the Navier–Stokes equations which read: given a bounded domain $\Omega \subset \mathbb{R}^3$ with a piecewise smooth boundary $\partial\Omega$, the simulation time T, and a force field $\mathbf{f}$: $(0,T]\times\Omega \to \mathbb{R}^3$, find a velocity field $\mathbf{u}$: $(0,T]\times\Omega \to \mathbb{R}^3$ and a pressure field p: $(0,T]\times\Omega \to \mathbb{R}$ such that

$$\begin{aligned} \partial_t \mathbf{u} - 2\nu\nabla\cdot\mathbb{D}\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in } (0,T]\times\Omega, \\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in } [0,T]\times\Omega, \\ \mathbf{u}|_{t=0} &= \mathbf{u}_0 \quad \text{in } \Omega, \end{aligned} \tag{1}$$

where $\nu>0$ is the kinematic viscosity coefficient and $\mathbb{D}\mathbf{u} := \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ denotes the velocity deformation tensor. Some boundary conditions have to be imposed on $\partial\Omega$ to obtain a closed set of equations. In the analysis below, we impose homogeneous Dirichlet conditions for simplicity, but see Remark 2.1.

In many industrial applications, simulations of turbulent flows are of major interest. Such flows are characterized by large Reynolds numbers $Re = UL/\nu$ with given characteristic length L and velocity scale U. For the numerical approximation, the finite element (FE) method is one of the most popular and mathematically sound variants. The standard Galerkin method aims to simulate all persistent scales in the range of order $\mathcal{O}(\text{diam}(\Omega))$ down to $\mathcal{O}(Re^{-3/4})$ which is not feasible even in next futures for the case of large Re. Residual-based stabilization techniques, like the streamline-upwind (or SUPG) method and/or the pressure stabilization (or PSPG) technique, add numerical viscosity acting at all scales. For a representative overview, we refer to Ref. [29]. The natural approach to simulate only the behaviour of large scales accurately has been considered for a long time in the classical Large-Eddy simulation (LES). Several drawbacks like commutation errors and the unsolved question of appropriate boundary conditions for the large scales have been critically discussed in recent times. For a review, see Ref. [4].

Based on ideas in Refs. [11,12], the class of variational multiscale (VMS) methods provides an alternative approach to the simulation of large scales. For a first application to turbulent flow problems, we refer to Ref. [13]. The basic idea of VMS methods is to define the large scales by projections into appropriate function spaces. Within a three-scale decomposition of the flow field into large, resolved small and unresolved scales, the influence of the unresolved small scales is described by a subgrid model acting directly only on the resolved small scales. A series of numerical studies reports good experience with VMS methods for standard benchmark problems. Meanwhile, different variants of VMS methods have been considered, for a review and comparison of different variants see Refs. [10,17].

The numerical analysis of VMS methods for turbulent flows is still in its infancy. Let us remark that the analysis depends on the choice of the discrete velocity–pressure approximation. For the case of equal-order interpolation, we refer to the contributions of Codina and his co-workers, see, e.g., Ref. [26]. The analysis may differ in certain aspects from the equal-order case if inf–sup stable FE pairs are applied (as in the present paper). Some progress has been made by John and co-

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workers for projection-based variants of the VMS method with inf–sup stable FE pairs. A globally constant turbulent viscosity ν_T together with an elliptic projection for the definition of the large scales had been used in Ref. [14] and analyzed in Ref. [15]. The recent paper [16] analyzes a Smagorinsky-type subgrid model applied together with an L^2-projection. The latter approach avoids some open problems of the elliptic projection in Ref. [14]. For a discussion on subgrid modeling of the unresolved pressure scales based on grad–div stabilization, we refer to Ref. [25].

In the present paper, we consider a modified projection-based FE-VMS method which had been presented in Refs. [14,20]. The subgrid model for the unresolved velocity scales is based on the L^2-projection Π_H for the definition of the large scales of the velocity deformation tensor. One difference to the approach in Ref. [16] is that the so-called fluctuation operator $I-\Pi_H$ is applied to the velocity deformation tensor whereas the velocity deformation tensor is applied to the fluctuation operator in Ref. [16] first. Please notice that these operators do not commute in the general case [24]. Another difference to Refs. [15,16] is the application of the so-called grad–div stabilization as a subgrid model for the pressure. In particular, we address implementation issues and the relation of the method to stabilization techniques based on local projection. We derive an a priori error estimate for the semidiscrete problem where the definition of the subgrid models for the unresolved velocity and pressure scales remains rather general. For the case of homogeneous isotropic turbulence, we specify the velocity subgrid model to be of Smagorinsky-type. In particular, Lilly's argument [13,22] is modified for this model. The subgrid parametrization includes the dependence on the polynomial degree of the velocity approximation. Finally, the parametrization of the two subgrid models is checked for the case of decaying homogeneous isotropic turbulence.

The paper is organized as follows: in Section 2, we introduce the projection-based VMS method under consideration. Then, in Section 3, we provide the error analysis for the model after spatial semidiscretization based on inf–sup stable finite element pairs for velocity and pressure. In Section 4 we specify the subgrid model for the case of homogeneous isotropic turbulence. Then, Section 5 is devoted to the application of the approach to the standard benchmark of decaying homogeneous isotropic turbulence. Finally, we summarize the results in Section 6 and give some conclusions.

2. A modified projection-based finite element variational multiscale method

2.1. Preliminaries

Let $\Omega\subset\mathbb{R}^3$ be a bounded domain. Standard notations are used for Lebesgues spaces $L^p(\Omega)$ and Sobolev spaces $W^{m,p}(\Omega)$, $m\in\mathbb{N}$, $1\le p\le\infty$, together with the corresponding norms $||\cdot||_{L^p(\Omega)}$ and $||\cdot||_{W^{m,p}(\Omega)}$. The inner product in $[L^2(\Omega)]^3$ will be denoted by $(\cdot,\cdot)$. A similar notation will be used on subdomains $D\subseteq\Omega$. For clarity we write $||\cdot||_0$ for the L^2 norm $||\cdot||_{L^2(\Omega)}$ of the whole domain Ω.

For a normed space X with functions defined on Ω, let $L^p(0,t;X)$ be the space of all functions defined on $(0,t)\times X$ with finite norm

$$\|\mathbf{u}\|_{L^p(0,t;X)}:=\left(\int_0^t\|\mathbf{u}\|_X^p ds\right)^{1/p},\ 1\le p<\infty$$

and with the obvious modification for $p=\infty$.

Setting $V=[H_0^1(\Omega)]^3$ and $Q=L_*^2(\Omega):=\{q\in L^2(\Omega):\int_\Omega q\,dx=0\}$, we consider the variational formulation of the Navier–Stokes equations: find $\mathbf{u}$: $[0,T]\to V$ and p: $(0,T]\to Q$ satisfying

$$\begin{aligned}(\partial_t\mathbf{u},\mathbf{v})+(2\nu\mathbb{D}\mathbf{u},\mathbb{D}\mathbf{v})+b_S(\mathbf{u},\mathbf{u},\mathbf{v})-(p,\nabla\cdot\mathbf{v})&=(\mathbf{f},\mathbf{v}) &&\forall\mathbf{v}\in V,\\ (q,\nabla\cdot\mathbf{u})&=0 &&\forall q\in Q.\end{aligned}\quad(2)$$

Here, the skew-symmetric trilinear form $b_S(\mathbf{u},\mathbf{v},\mathbf{w}):=\frac{1}{2}[((\mathbf{u}\cdot\nabla)\mathbf{v},\mathbf{w})-((\mathbf{u}\cdot\nabla)\mathbf{w},\mathbf{v})]$ has the important property $b_S(\mathbf{u},\mathbf{v},\mathbf{v})=0$ for all $\mathbf{u},\mathbf{v}\in V$.

For the present analysis, we will use Korn's inequality with constant C_{Ko} and the Poincaré–Friedrichs inequality with constant C_F such that

$$||\nabla\mathbf{v}||_0\le C_{Ko}||\mathbb{D}\mathbf{v}||_0\quad\text{and}\quad||\mathbf{v}||_0\le C_F||\nabla\mathbf{v}||_0\quad\forall\mathbf{v}\in V.\quad(3)$$

Remark 2.1. The analysis of this paper can be applied in the case of periodic boundary conditions for the velocity as well. The proof for Korn's inequality under such conditions is very similar to the case of no-slip boundary conditions, see Ref. [28]. Please notice that the application of the VMS approach to the case of decaying homogeneous isotropic turbulence requires such periodic boundary conditions. It seems also possible to extend the analysis to cases where no-slip boundary conditions and periodic boundary conditions appear simultaneously, e.g. in channel flows.

2.2. Variational multiscale method

Let $\mathcal{T}_h$ be an admissible triangulation of Ω in the usual sense, see Ref. [8], with maximal diameter $h>0$ of the mesh cells $K\in\mathcal{T}_h$. The FE spaces $V_h\times Q_h\subset V\times Q$ of the basic Galerkin FE method will be standard inf–sup stable velocity–pressure spaces, i.e. with

$$\inf_{q_h\in Q_h\setminus\{0\}}\ \sup_{\mathbf{v}_h\in V_h\setminus\{0\}}\frac{(q_h,\nabla\cdot\mathbf{v_h})}{||q_h||_0||\nabla\mathbf{v}_h||_0}\ge\beta>0\quad(4)$$

where β is h-independent. The Galerkin FE method reads: find $\mathbf{u}_h$: $[0,T]\to V_h$, p_h: $(0,T]\to Q_h$ such that

$$\begin{aligned}(\partial_t\mathbf{u}_h,\mathbf{v}_h)+(2\nu\mathbb{D}\mathbf{u}_h,\mathbb{D}\mathbf{v}_h)+b_S(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h)-(p_h,\nabla\cdot\mathbf{v}_h)&=(\mathbf{f},\mathbf{v}_h) &&\forall\mathbf{v}_h\in V_h,\\ (q_h,\nabla\cdot\mathbf{u}_h)&=0 &&\forall q_h\in Q_h.\end{aligned}$$

For turbulent flows, let a three-scale decomposition of the flow and pressure fields be given by

$$\mathbf{v}=\overline{\mathbf{v}}_h+\tilde{\mathbf{v}}_h+\hat{\mathbf{v}}\quad\forall\mathbf{v}\in V;\quad q=\overline{q}_h+\tilde{q}_h+\hat{q}\quad\forall q\in Q.$$

We search for the resolved scales $(\mathbf{v}_h,q_h):=(\overline{\mathbf{v}}_h+\tilde{\mathbf{v}}_h,q_h+\tilde{q}_h)\in V_h\times Q_h\subset V\times Q$. The influence of the unresolved small velocity scales $(\hat{\mathbf{v}}_h,\hat{q}_h)$ on the resolved small scales will be modeled using a variant of the variational multiscale approach of Ref. [20], Section 3. To this goal, we define the following.

Definition 2.2. Let $\mathcal{T}_H$ be the triangulation of a coarser grid, i.e. $H\ge h$. Then the finite element space L_H of coarse scales of the deformation tensor is

$$\{0\}\subseteq L_H\subseteq\mathbb{D}V_h\subseteq L:=\left\{\mathbf{L}=\left(l_{ij}\right)|\,l_{ij}=l_{ji}\in L^2(\Omega)\forall i,j\in\{1,2,3\}\right\}.$$

Within this article we assume that $\mathcal{T}_h$ is a conforming refinement of $\mathcal{T}_H$. A possible choice of the space L_H with $H=h$ will be discussed in Section 4. In particular, an adaptive choice of L_H, which varies on $\mathcal{T}_H$, is not excluded, see Ref. [17].

A first version of the VMS model reads: find $\mathbf{u}_h$: $[0,T]\to V_h$, p_h: $(0,T]\to Q_h$, $\mathbf{G}_H$: $(0,T]\to L_H$ such that

$$\begin{aligned}(\partial_t\mathbf{u}_h,\mathbf{v}_h)+(2\nu\mathbb{D}\mathbf{u}_h,\mathbb{D}\mathbf{v}_h)+b_S(\mathbf{u}_h,\mathbf{u}_h,\mathbf{v}_h)-(p_h,\nabla\cdot\mathbf{v}_h)&\\ +(\nu_T(\mathbf{u}_h)(\mathbb{D}\mathbf{u}_h-\mathbf{G}_H),\mathbb{D}\mathbf{v}_h)&=(\mathbf{f},\mathbf{v}_h) &&\forall\mathbf{v}_h\in V_h,\\ (q_h,\nabla\cdot\mathbf{u}_h)&=0 &&\forall q_h\in Q_h,\\ (\mathbf{G}_H-\mathbb{D}\mathbf{u}_h,\mathbf{L}_H)&=0 &&\forall\mathbf{L}_H\in L_H.\end{aligned}$$