Computer Methods in Applied Mechanics and Engineering 199 (2010) 2487–2496

Contents lists available at ScienceDirect

Computer Methods in Applied Mechanics and Engineering

journal homepage: www.elsevier.com/locate/cma

Adaptive element-free Galerkin method applied to the limit analysis of plates

Canh V. Le, Harm Askes *, Matthew Gilbert

University of Sheffield, Department of Civil and Structural Engineering, Sheffield S1 3JD, United Kingdom

ARTICLE INFO

Article history:
Received 2 March 2009
Received in revised form 18 March 2010
Accepted 12 April 2010
Available online 18 April 2010

Keywords:
Error estimation
Adaptivity
Limit analysis
EFG method
Second-order cone programming

ABSTRACT

The implementation of an *h*-adaptive element-free Galerkin (EFG) method in the framework of limit analysis is described. The naturally conforming property of meshfree approximations (with no nodal connectivity required) facilitates the implementation of *h*-adaptivity. Nodes may be moved, discarded or introduced without the need for complex manipulation of the data structures involved. With the use of the Taylor expansion technique, the error in the computed displacement field and its derivatives can be estimated throughout the problem domain with high accuracy. A stabilized conforming nodal integration scheme is extended for use in error estimation and results in an efficient and truly meshfree adaptive method. To demonstrate its effectiveness the procedure is then applied to plates with various boundary conditions.

1. Introduction

Limit analysis makes use of the fundamental theorems of plastic analysis to provide a powerful means of estimating the maximum load sustainable by a solid or structure. Mathematical programming techniques can often be applied to permit the collapse load to be determined directly. In recent years efforts have focussed principally on the development of efficient and robust numerical limit analysis tools of potential use to engineers working in practice, which either use continuous (e.g. using finite element [1–7] or meshless [8–10] methods), semi-continuous [11] or truly discontinuous [12] representations of the relevant field variables. However, the accuracy of numerical limit analysis solutions is highly affected by local singularities arising from localized plastic deformations [13]. In order to achieve accurate solutions automatic *h*-refinement is often performed, so that the resolution of the spatial discretization is refined in plastic zones. Automatic finite element mesh refinement based on both stress and strain fields has been previously proposed [14], where elements are candidates for refinement if the strain tensor is non-zero. Alternatively, adaptive procedures based on *a posteriori* error estimates to measure the local and global errors associated with the interpolation have been developed for limit analysis problems. A directional error estimate using recovery gradients and/or the Hessian of mixed finite element solutions was proposed in [13]. The scheme was then adapted to lower bound limit analysis by using quasi-velocities and plastic multipliers from the dual solution [15]. Using solutions of the lower and upper bound problem in combination, another effective error estimate was proposed in [16,17]. These techniques have been used successfully for various 2D problems.

Meshfree methods are very attractive computational techniques due to their flexibility, e.g. no nodal connectivity is required. The naturally conforming property of meshfree approximations offers considerable advantages in adaptive analysis. Nodes can easily be added and removed without the need for complex manipulation of the data structures involved. Since error estimates for finite elements are not always directly transferable to meshfree methods, various approaches have been proposed [18–23]. Effective approaches to estimate the interpolation/approximation error were proposed in [20,21,24,25]. The approximation error in the computed displacement field and its derivatives can be evaluated with high accuracy using a Taylor expansion of the relevant field variable. It is also shown in [21] that this estimate is generally suitable for problems with high stress and strain gradients and singularities. While these approaches have been developed for structured meshfree particle methods using Gauss integration, it is also desirable to develop an efficient method for general irregular nodal layouts. In this paper the error density in a representative nodal cell can be determined using smoothed values of the displacement derivatives. This not only results in a truly meshfree method but also reduces the effort required to calculate displacement derivatives in the error estimate. Furthermore, since the Voronoi diagram for a set of nodes is unique, properties of Voronoi cells can be conveniently used as a reference for refinement strategies and for determining locally the size of the domain of influence.

In the framework of meshfree methods, it is advantageous if the problem under consideration can be solved by evaluating quantities at the nodes only [26–32]. The smoothing technique proposed in [28] is one of the most efficient nodal integration methods available, and has been applied successfully to various analysis problems [9,10,33–35].

* Corresponding author.
E-mail addresses: Canh.Le@sheffield.ac.uk (C.V. Le), h.askes@sheffield.ac.uk (H. Askes), m.gilbert@sheffield.ac.uk (M. Gilbert).

0045-7825/$ – see front matter © 2010 Elsevier B.V. All rights reserved.
doi:10.1016/j.cma.2010.04.004

In this technique nodal values are determined by spatially averaging field values using the divergence theorem. In other words, the domain integrals are transformed into boundary integrals to avoid the evaluation of the derivatives of the meshfree shape functions at the nodes, where they vanish, thus eliminating spatial instability problems. In this paper the range of applicability of the smoothing technique is extended by applying it to a kinematic limit analysis formulation incorporating error estimation.

The objective of this paper is to develop a meshfree h-adaptivity procedure for limit analysis problems. The layout of the paper is as follows: Section 2 briefly describes a kinematic upper-bound limit analysis formulation for plates using the element-free Galerkin (EFG) method and stabilized conforming nodal integration (SCNI). A cell-based error estimate for the displacement field and its derivatives is presented in Section 3. Based on the error estimate discussed in Section 3, error indicators and refinement strategies are introduced in Section 4. Numerical examples are provided in Section 5 to illustrate the performance of the proposed procedure.

2. Limit analysis of plates — discrete kinematic formulation

In this section the kinematic formulation for the plate limit analysis problem is outlined, together with details of the EFG method and the second-order cone programming (SOCP) problem formulation. More details can be found in [9].

Consider a rigid-perfectly plastic plate governed by the von Mises yield criterion, subjected to a distributed load $\alpha^+ q$ and with a constrained boundary Γ_u. The upper-bound limit analysis problem for plates can be written as

$$\alpha^+ = \min \int_\Omega m_p \|\mathbf{C}^T \dot{\boldsymbol{\kappa}}\|_{L^2(\Omega)} d\Omega \quad \text{s.t} \tag{1a}$$

$$\dot{\boldsymbol{\kappa}} = -\left\{ \frac{\partial^2 \dot{u}^h}{\partial x^2} \quad \frac{\partial^2 \dot{u}^h}{\partial y^2} \quad 2\frac{\partial^2 \dot{u}^h}{\partial x \partial y} \right\}^T \tag{1b}$$

$$\int_\Omega q u^h \, d\Omega = 1 \tag{1c}$$

accompanied by appropriate boundary conditions, where q is unit load per area, α^+ is a scalar collapse load multiplier, $m_p = \sigma_0 t^2/4$ is the plastic moment of resistance per unit width of a plate of thickness t and $\mathbf{C}$ is a matrix that depends on the yield criterion involved. For the von Mises criterion,

$$\mathbf{C} = \frac{1}{\sqrt{3}} \begin{bmatrix} 2 & 0 & 0 \\ 1 & \sqrt{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

The approximated transverse displacement $u^h(\mathbf{x})$ is computed using a Moving Least Squares (MLS) technique and is expressed as

$$u^h(\mathbf{x}) = \sum_{I=1}^{n} \Phi_I(\mathbf{x}) u_I \tag{3}$$

The MLS shape functions $\Phi_I(\mathbf{x})$ are given as [36,37]

$$\Phi_I(\mathbf{x}) = \mathbf{p}^T(\mathbf{x}) \mathbf{A}^{-1}(\mathbf{x}) \mathbf{B}_I(\mathbf{x}) \tag{4}$$

with

$$\mathbf{A}(\mathbf{x}) = \sum_{I=1}^{n} w_I(\mathbf{x}) \mathbf{p}(\mathbf{x}_I) \mathbf{p}^T(\mathbf{x}_I) \tag{5}$$

$$\mathbf{B}_I(\mathbf{x}) = w_I(\mathbf{x}) \mathbf{p}(\mathbf{x}_I) \tag{6}$$

where n is the number of nodes; $\mathbf{p}(\mathbf{x}) = [1,\ x,\ y,\ xy,\ x^2,\ y^2]^T$ is a quadratic basis function and $w_I(\mathbf{x})$ is an isotropic quartic spline weight function associated with node I.

Introducing stabilized conforming nodal integration [28], smoothed curvature rates $\dot{\boldsymbol{\kappa}}(\mathbf{x}_j)$ at nodal point $\mathbf{x}_j$ are written as

$$\dot{\boldsymbol{\kappa}}(\mathbf{x}_j) = -\mathbf{G}\,\mathbf{v} \tag{7}$$

where

$$\mathbf{v}^T = [\dot{u}_1, \dot{u}_2, \ldots, \dot{u}_n] \tag{8}$$

$$\mathbf{G} = \begin{bmatrix} \tilde{\Phi}_{1,xx}(\mathbf{x}_j) & \tilde{\Phi}_{2,xx}(\mathbf{x}_j) & \ldots & \tilde{\Phi}_{n,xx}(\mathbf{x}_j) \\ \tilde{\Phi}_{1,yy}(\mathbf{x}_j) & \tilde{\Phi}_{2,yy}(\mathbf{x}_j) & \ldots & \tilde{\Phi}_{n,yy}(\mathbf{x}_j) \\ 2\tilde{\Phi}_{1,xy}(\mathbf{x}_j) & 2\tilde{\Phi}_{2,xy}(\mathbf{x}_j) & \ldots & 2\tilde{\Phi}_{n,xy}(\mathbf{x}_j) \end{bmatrix} \tag{9}$$

with

$$\begin{aligned} \tilde{\Phi}_{I,\alpha\beta}(\mathbf{x}_j) &= \frac{1}{2a_j} \oint_{\Gamma_j} \left(\Phi_{I,\alpha}(\mathbf{x}_j) n_\beta(\mathbf{x}) + \Phi_{I,\beta}(\mathbf{x}_j) n_\alpha(\mathbf{x}) \right) d\Gamma \\ &= \frac{1}{4a_j} \sum_{k=1}^{ns} \left(n_\beta^k l^k + n_\beta^{k+1} l^{k+1} \right) \Phi_{I,\alpha}(\mathbf{x}_j^{k+1}) \\ &\quad + \frac{1}{4a_j} \sum_{k=1}^{ns} \left(n_\alpha^k l^k + n_\alpha^{k+1} l^{k+1} \right) \Phi_{I,\beta}(\mathbf{x}_j^{k+1}) \end{aligned} \tag{10}$$

where $\tilde{\boldsymbol{\Phi}}$ is the smoothed version of $\boldsymbol{\Phi}$; a_j, Γ_j and ns are respectively the area, boundary and the number of segments of a Voronoi nodal domain Ω_j as shown in Fig. 1; $\mathbf{x}_j^k$ and $\mathbf{x}_j^{k+1}$ are the coordinates of the two end points of boundary segment Γ_j^k which has length l^k and outward surface normal n^k.

The kinematic limit analysis problem for plates can now be written in the form of a SOCP problem as follows:

$$\alpha^+ = \min \ m_p \sum_{j=1}^{n} a_j t_j \quad \text{s.t} \tag{11a}$$

$$\mathbf{A}_{eq} \mathbf{v} = \mathbf{b}_{eq} \tag{11b}$$

$$-\mathbf{C}^T \mathbf{G} \mathbf{v} = \mathbf{r}_i \tag{11c}$$

$$\|\mathbf{r}_i\| \le t_i, \quad i = 1, 2, \ldots, n \tag{11d}$$

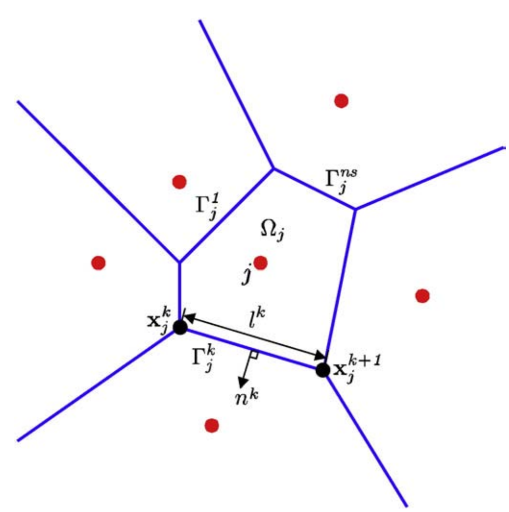

Fig. 1. Geometry of a representative nodal domain.