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A class of discontinuous Petrov–Galerkin methods. Part I: The transport equation

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ABSTRACT

Considering a simple model transport problem, we present a new finite element method. While the new method fits in the class of discontinuous Galerkin (DG) methods, it differs from standard DG and streamline diffusion methods, in that it uses a space of discontinuous trial functions tailored for stability. The new method, unlike the older approaches, yields optimal estimates for the primal variable in both the element size h and polynomial degree p, and outperforms the standard upwind DG method.

1. Introduction

We introduce a new Petrov–Galerkin method for advective problems. While it belongs in the class of discontinuous Galerkin (DG) methods, unlike the standard upwind DG method, we are able to prove *optimal h and p error estimates* in the L^2-norm for our discrete solution on general meshes (where, as usual, h is the mesh size and p is the polynomial degree). The method includes a separate outflux approximation on element interfaces and a space of non-standard test functions designed for stability.

The boundary value problem that is the subject of this paper is posed on a polyhedral domain Ω. Given f and g, we need to find a finite element approximation to the solution u of

$$\vec{\beta} \cdot \vec{\nabla} u = f \quad \text{on } \Omega, \tag{1a}$$
$$u = g \quad \text{on } \partial_{\text{in}}\Omega. \tag{1b}$$

We only consider the case of constant $\vec{\beta}$ in this paper (but extensions are possible, as mentioned in Section 5). The inflow boundary $\partial_{\text{in}}\Omega$ appearing in (1) is defined, letting $\vec{n}$ denote the unit outward normal, by

$$\partial_{\text{in}}\Omega = \{\vec{x} \in \partial\Omega : \vec{\beta} \cdot \vec{n}(\vec{x}) < 0\}, \tag{2}$$

i.e., $\partial_{\text{in}}\Omega$ denotes the global inflow boundary. While non-finite-element numerical techniques can be designed for this problem (e.g. the method of characteristics), we aim for finite elements because of its versatility in handling complicated domains as well as certain regular and singular perturbations of the above problem. A regular perturbation of (1) is

$$\vec{\beta} \cdot \vec{\nabla} u + \alpha(\vec{x}) u = f. \tag{3}$$

A singular perturbation of (1) is obtained by the addition of a small viscosity term with second derivatives. This is harder to analyze. Within the domain of finite element methods for (1), there are two broad categories (see [11] for a review). One is the very popular streamline diffusion method [13] and its descendants. The other category is composed of DG methods. Since our contribution fits in the latter, we shall now review previous works in this category in detail.

The well known first papers proposing and analyzing the original DG method for (1) are [14,15,18]. To distinguish this method

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from our DG method, we will call the original DG method the "upwind DG method" and denote it by UDG, while we call ours the "discontinuous Petrov–Galerkin method" and denote it by DPG. It is proved in [15] that if u_h is the UDG approximation, then (for a fixed p) it satisfies $\|u - u_h\|_{L^2(\Omega)} \leqslant O(h^{s-1})$ for some $s \leqslant p+1$ dictated by the regularity of the exact solution. This result was improved by [14] wherein it was shown that the rate of convergence is in fact $O(h^{s-1/2})$. In both cases convergence with respect to p was not studied. Even if we set aside the p-convergence issue, notice that both the results are suboptimal in h, as the best approximation error of the finite element space is $O(h^s)$.

For some special classes of meshes however, many authors have observed (and proved) the optimal rate of convergence of the UDG method [8,19] with respect to h. Nonetheless, on general meshes, the suboptimal rate of convergence cannot be improved, as shown by a numerical example in [17] using a particular quasiuniform mesh and a smooth exact solution. To express the sentiment of many, we quote from [8] that "the mechanisms that induce the loss of $h^{1/2}$ in the order of convergence of the L^2-norm of the error are not very well known yet".

An *hp* analysis of the UDG scheme was first provided in [3]. They considered the regular perturbation (3) under the assumption that

$$0 < c_0 \leqslant \alpha(\vec{x}) \quad \forall \vec{x} \in \Omega. \tag{4}$$

Because of this assumption, they are able to control the $L^2(\Omega)$-norm of the solution. They also introduced a stabilization parameter into the original upwind DG method. A few years later, the paper [12] extended the results of [3] in several directions, providing a unified theory for an *hp* version of the streamline diffusion method, as well as the upwind DG method. Their analysis did not assume (4), rather they let the advection vector $\vec{\beta}$ depend on $\vec{x}$ and assumed

$$0 < c_0 \leqslant -\frac{1}{2}\nabla \cdot \beta(\vec{x}) + \alpha(\vec{x}) \quad \forall \vec{x} \in \Omega, \tag{5}$$

as a consequence of which they have stability in $c_0\|u\|_{L^2(\Omega)}$. (Note that for the case we intend to study (1), the right hand side of (5) evaluates to zero, so (5) does not hold.) Both papers analyzed the stabilized version of the upwind DG method and both relied on the proper choice of the stabilization parameter. While the results of [12] are optimal in p, those of [3] are suboptimal in p. In all these works, the $L^2(\Omega)$-rate of convergence of the error with respect to h remained suboptimal by $h^{1/2}$. In contrast, our results do not exhibit this suboptimality, nor do we add any stabilization parameter. We believe our method is the first in the finite element (not just DG) family of methods for the transport equation which has provably optimal convergence rates on very general meshes.

The design of our method is guided by a generalization of Céa Lemma due to Babuška [1,5]. We only need a simple version of the result, which we now describe using the following notations (all our spaces are over $\mathbb{R}$). Let X, $X_h \subset X$, and V_h be Banach spaces and let $a_h(\cdot,\cdot)$ be a bilinear form on $X \times V_h$. Suppose the exact solution $U \in X$ satisfies and the discrete solution $U_h \in X_h$ satisfies

$$a_h(U - U_h, v_h) = 0 \quad \text{for all } v_h \in V_h. \tag{6}$$

If the bilinear form is continuous in the sense that there is a $C_1 > 0$ such that

$$a_h(w, v_h) \leqslant C_1 \|w\|_X \|v_h\|_{V_h} \quad \text{for all } w \in X, \;\; v_h \in V_h \tag{7}$$

and also the inf–sup condition, i.e., there is a $C_2 > 0$ such that

$$C_2 \|w_h\|_X \leqslant \sup_{v_h \in V_h} \frac{a_h(w_h, v_h)}{\|v_h\|_{V_h}} \quad \text{for all } w_h \in X_h, \tag{8}$$

then, as is well known, the following theorem can be formulated:

Theorem 1.1. *Under the above setting, we have the following error estimate:*

$$\|U - U_h\|_X \leqslant \left(1 + \frac{C_1}{C_2}\right) \inf_{w_h \in X_h} \|U - w_h\|_X.$$

Proof. The argument is simple and standard:

$$\begin{aligned}
\|U - U_h\|_X &= \|U - w_h\|_X + \|w_h - U_h\|_X \\
&\leqslant \|U - w_h\|_X + \frac{1}{C_2} \sup_{v_h \in V_h} \frac{a_h(w_h - U_h, v_h)}{\|v_h\|_{V_h}} \quad \text{by (8)}, \\
&\leqslant \|U - w_h\|_X + \frac{1}{C_2} \sup_{v_h \in V_h} \frac{a_h(w_h - U, v_h)}{\|v_h\|_{V_h}} \quad \text{by (6)}, \\
&\leqslant \|U - w_h\|_X + \frac{C_1}{C_2} \|w_h - U\|_X \quad \text{by (7)}.
\end{aligned}$$

This finishes the proof. □

Many refinements and improvements of such a theorem are known. But our purpose in going through the above simple argument is to clearly show that the test space need not have approximation properties. Hence, in designing Petrov–Galerkin methods, *while we must choose trial spaces with good approximation properties, we may design test spaces solely to obtain good stability properties*. This will be our guiding principle in designing our method. In fact, the test spaces we propose shortly can have discontinuities *inside* the mesh elements.

Many researchers have put the above principle to good use. In fact, even the abbreviation we use for our new method "DPG method", has been previously used [4,6] for other methods. The theme in these works is the search for stable test spaces using bubbles or other polynomials. Our test space functions, in contrast, need not be polynomial on an element, and indeed, need not even be continuous. We are also not the first to consider such functions with discontinuities within a finite element. Such elements are routinely used in X-FEM and similar methods [2] for difficult simulations like crack propagation. However, we use discontinuities solely for stability purposes, and solely in test spaces. Our trial spaces, being standard polynomial spaces, possess provably good approximation properties.

Our method also introduces a new flux unknown on the element interfaces. This is in line with the recent developments on hybridized DG (HDG) methods [9]. HDG methods that extend the ideas in [9] to the case of convection can be found in recent works [10,16]. These methods are constructed by defining a independent flux variable on the element interfaces which can solved for first, after which the internal variables can be locally solved for. While this can be thought of as akin to static condensation, additional advantages can be exploited, such as easy stabilization [16] using a penalty parameter. However, p-independent stability for such methods has not been proved yet. While we borrow the idea of letting the fluxes be independent variables in the design of our method, our method does not have stabilization parameters, and has p-independent stability.

We organize our presentation such that a spectral version of the method is first exhibited (in Section 2). Details regarding the new space of test functions and the stability estimates for the method on a single element are presented in that section. Section 3 then presents the composite method on a triangular mesh. Optimal L^2 error estimates are proved in Theorem 3.2 there. We conclude in Section 5 opining on important future directions. Proofs of a few technical estimates are gathered in Appendix A.

2. The spectral method on one element

We start by considering the one-element case to fix the ideas and study the element spaces. In other words, we let Ω be an