Comput. Methods Appl. Mech. Engrg. 199 (2010) 791–801

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

The dissipative structure of variational multiscale methods for incompressible flows

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ARTICLE INFO

Article history:
Received 16 January 2008
Received in revised form 25 July 2008
Accepted 4 September 2008
Available online 24 September 2008

Keywords:
Stabilized finite elements
Transient subscales
Orthogonal subscales
Energy transfer

ABSTRACT

In this paper, we present a precise definition of the numerical dissipation for the orthogonal projection version of the variational multiscale method for incompressible flows. We show that, only if the space of subscales is taken orthogonal to the finite element space, this definition is physically reasonable as the coarse and fine scales are properly separated. Then we compare the diffusion introduced by the numerical discretization of the problem with the diffusion introduced by a large eddy simulation model. Results for the flow around a surface-mounted obstacle problem show that numerical dissipation is of the same order as the subgrid dissipation introduced by the Smagorinsky model. Finally, when transient subscales are considered, the model is able to predict backscatter, something that is only possible when *dynamic* LES closures are used. Numerical evidence supporting this point is also presented.

1. Introduction

Variational multiscale (VMS) finite element methods for large eddy simulation (LES) of turbulence are promising. First attempts to apply the VMS idea to incompressible flow problems were made in [17,18], where both small and large scales were solved and the classical LES filters were applied only to the small scales. The idea of simply using the algebraic approximation of the subscales, without any additional ingredient (e.g. without physical-based subgrid modeling), was introduced in [7] and elaborated in [5,19]. Very good results were obtained for fully developed and transitional turbulent flows. For a complete presentation in the context of isogeometric analysis, including results of homogeneous turbulence and turbulent channel flow, we refer the reader to [2]. It is important to note, however, that the algebraic approximation to the subscales in [7,5,19] and in this work includes terms additional to those appearing in the classical GLS/SUPG methods [5,9].

Many comments have been and are usually made on the importance of the numerical scheme when a LES of a turbulent flow is performed. Chapter 7 of [26] is devoted to the numerical solution of the LES equations and several results are mentioned. The influence of the numerical scheme and its interaction with classical LES models was analyzed in [12], where truncation errors are compared with the amplitude of the subgrid terms and found to be dominant in many cases. The solution suggested in [12] is either to increase the accuracy of the scheme or to use the "pre-filtering" technique (to keep the filter size constant while the mesh size is reduced until h-convergence is achieved). As mentioned in [26], numerical experiments presented in [21] show that "the effect of subgrid models is completely or partially masked by the numerical error when second-order accurate methods are employed". The use of high order accurate schemes in order to minimize numerical dissipation is not an uncommon advice. It is also quite common to describe a numerical method according to how dissipative it is. However, precise measures of this property have not been presented up to date.

The first attempt to estimate numerical and subgrid dissipation was made in [11], where several schemes for LES of compressible flows are compared. The numerical dissipation is linked to the leading terms of the truncation error and an intuitive definition is presented. This definition is based on the difference between the discrete convective term and that given by a reference centered scheme of one order of accuracy higher. A scheme is considered suitable for LES if either the numerical dissipation is much lower than the subgrid one (condition C1) or the numerical dissipation is able to mimic the subgrid one (condition C2). The general sad conclusion is that neither condition C1 nor condition C2 are satisfied for the schemes analyzed.

Further analyses have been made in [10], where a method to compute the effective numerical dissipation is developed based on a finite difference approximation of the energy balance equation. The method is used to evaluate the dissipation in the context of the monotonically integrated LES approach, first proposed in [3], in which the Navier–Stokes equations are directly discretized without introducing neither the filtering operation nor the SGS stress tensor. The objective in [10] is to link the properties of the numerical method with the physics of turbulence by comparing numerical dissipation with the predictions of turbulence theory.

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doi:10.1016/j.cma.2008.09.007

A Galerkin least squares multiscale formulation of compressible flows was presented in [23]. In this reference the original multiscale formulation of LES introduced in [17] is advocated and the small scales are computed and filtered. The numerical dissipation introduced by the least squares terms, obtained multiplying the stabilization terms in the final algebraic system by the unknown, is greater than the subgrid dissipation for the closures analyzed. A similar approach is followed in [28] and the global conservation of energy is used to define the numerical dissipation introduced by the SUPG scheme (even the SUPG stress, an analog to the subgrid stress, is defined). A dynamic Smagorinsky model is used and the interaction with the dissipation introduced by the SUPG scheme is analyzed. Results of channel flow simulations show the numerical dissipation to be smaller than the physical one but still important. These results are used to propose a correction to the physical model to take into account the numerical scheme. The basic idea is that, ideally, the numerical scheme should not affect the physical model and therefore the proposal of [28] is to modify the physical model reducing the dissipation it introduces by the same amount the SUPG dissipation provides. This approach is opposed to that advocated in [3] and in the present work.

The *global* energy balance of the finite element component is also the starting point in [4] where the interior penalty method is analyzed and the quality of the solution is evaluated in terms of the numerical dissipation. In [13] it is shown that the numerical dissipation provided by the orthogonal subgrid scale method of Codina [7] provides a rate of transfer of subgrid kinetic energy proportional to the molecular physical dissipation rate (for an appropriate choice of the stabilization parameter), thus precluding in principle the need of introducing an extra LES physical model.

In this article we introduce a *local* definition of the numerical dissipation for the variational multiscale method based on the local version of the finite element energy balance. We also consider the energy balance of the subgrid scale component, which permits us to clearly identify the energy transfer mechanisms. In this framework we show that only the orthogonal subgrid scale method of Codina [7] permits a proper separation of scales, in the sense that if a non-orthogonal projection is used, temporal derivatives couple the energy balance for the coarse and fine scale components. When the time dependent subscales of [9] are used, the model is capable of predicting backscatter (energy transfer from small to coarse scales).

The paper is organized as follows. In Section 2 the problem is stated, and in Section 3 its two scale approximation introduced. The core of the paper is presented in Section 4, where the energy budget in a region is discussed. Section 5 presents the numerical simulation of the flow over a surface mounted obstacle, where the different dissipation mechanisms can be observed. Some final conclusions close the paper in Section 6.

2. Problem statement

Let us consider the flow of an incompressible fluid in a domain $\Omega \subset \mathbb{R}^d (d = 2, 3)$ with boundary $\Gamma = \partial\Omega$ during the time interval $[0, T]$. Let $\boldsymbol{u} : Q \to \mathbb{R}^d$ be the velocity field and $p : Q \to \mathbb{R}$ the pressure, with $Q = \Omega \times (0, T)$. The incompressible Navier–Stokes equations for $\boldsymbol{u}$ and p can be written as

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla u + \nabla p - \nu \nabla^2 \boldsymbol{u} = \boldsymbol{f}, \tag{1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2}$$

where $\boldsymbol{f}$ is the vector of external forces and ν is the kinematic viscosity. Eqs. (1) and (2) must be supplemented with appropriate boundary and initial conditions. For simplicity in the presentation, only homogeneous Dirichlet boundary conditions will be considered.

We will also consider the LES problem that is found by applying a filter of the form

$$\bar{\boldsymbol{v}}(\boldsymbol{x}) = \int \boldsymbol{v}(\boldsymbol{x}') G(\boldsymbol{x}, \boldsymbol{x}') \mathrm{d}\boldsymbol{x}',$$

to the Navier–Stokes Eqs. (1) and (2) for an appropriate filter function G. This operation results in an extra term: the divergence of the subgrid stress tensor. The LES problem consists in finding the filtered velocity field $\bar{\boldsymbol{u}}$ and the pressure field $\bar{\boldsymbol{p}}$ such that

$$\partial_t \bar{\boldsymbol{u}} + \bar{\boldsymbol{u}} \cdot \nabla \bar{\boldsymbol{u}} + \nabla \bar{\boldsymbol{p}} - \nu \nabla^2 \bar{\boldsymbol{u}} + \nabla \cdot \tau = \boldsymbol{f}, \tag{3}$$

$$\nabla \cdot \bar{\boldsymbol{u}} = 0, \tag{4}$$

where τ is the residual stress tensor defined in components by

$$\tau_{ij} = \overline{\boldsymbol{u}_i \boldsymbol{u}_j} - \bar{\boldsymbol{u}}_i \bar{\boldsymbol{u}}_j$$

Let $k = \frac{1}{2} \boldsymbol{u} \cdot \boldsymbol{u}$ be the pointwise kinetic energy. A kinetic energy conservation statement for the Navier–Stokes problem can be found by multiplying (1) by the velocity $\boldsymbol{u}$. Using (2) one obtains

$$\partial_t k + \boldsymbol{u} \cdot \nabla k + \boldsymbol{u} \cdot \nabla p - \nu \nabla^2 k + \nu \nabla u : \nabla u = \boldsymbol{u} \cdot \boldsymbol{f}.$$

This equation integrated over an arbitrary volume $\omega \subset \Omega$ and simplified by the use of (2) gives

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t} \int_\omega k}_{\mathrm{I}} + \underbrace{\int_{\partial\omega} \boldsymbol{n} \cdot [\boldsymbol{u}(k + p) - \nu \nabla k]}_{\mathrm{II}} + \underbrace{\int_\omega \nu \nabla u : \nabla u}_{\mathrm{III}} = \underbrace{\int_\omega \boldsymbol{u} \cdot \boldsymbol{f}}_{\mathrm{IV}}. \tag{5}$$

The total energy variation in volume ω (term I) is balanced by the flow of energy through its boundary (term II) plus the dissipation due to viscous effects (term III) and the work of external forces (term IV).

For the LES problem the kinetic energy conservation of the filtered velocity ($\bar{k} = \frac{1}{2} \bar{\boldsymbol{u}} \cdot \bar{\boldsymbol{u}}$) is found multiplying (3) by the filtered velocity $\bar{\boldsymbol{u}}$. In this case, the presence of the subgrid stresses gives rise to a term that is responsible for an energy exchange between large scales (represented by filtered variables) and small scales (represented by subgrid variables), which are given by $\boldsymbol{u}' = \boldsymbol{u} - \bar{\boldsymbol{u}}$ and $p' = p - \bar{\boldsymbol{p}}$. The result reads

$$\partial_t \bar{k} + \bar{\boldsymbol{u}} \cdot \nabla \bar{k} + \bar{\boldsymbol{u}} \cdot \nabla \bar{p} - \nu \nabla^2 \bar{k} + \nu \nabla \bar{\boldsymbol{u}} : \nabla \bar{\boldsymbol{u}} + \nabla \cdot (\bar{\boldsymbol{u}} \cdot \tau) + \nabla \bar{\boldsymbol{u}} : \tau = \bar{\boldsymbol{u}} \cdot \bar{\boldsymbol{f}}.$$

This equation integrated over an arbitrary volume ω and simplified by the use of (4) gives

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}t} \int_\omega \bar{k}}_{\mathrm{I}} + \underbrace{\int_{\partial\omega} \boldsymbol{n} \cdot [\bar{\boldsymbol{u}}(\bar{k} + \bar{\boldsymbol{p}}) - \nu \nabla \bar{k} + \bar{\boldsymbol{u}} \cdot \tau]}_{\mathrm{II}} + \underbrace{\int_\omega \nu \nabla \bar{\boldsymbol{u}} : \nabla \bar{\boldsymbol{u}}}_{\mathrm{III}} + \underbrace{\int_\omega \nabla \bar{\boldsymbol{u}} : \tau}_{\mathrm{V}} = \underbrace{\int_\omega \bar{\boldsymbol{u}} \cdot \bar{\boldsymbol{f}}}_{\mathrm{IV}}. \tag{6}$$

The interpretation of the terms corresponds to the meaning they had before, except for the new term V that represents the transfer of energy between coarse and fine scales and for the last term in II which now includes the flow of energy through the boundary due to the work done by the mean velocity against the residual stress tensor. Note that this term comes from the filtering of the convective term and when the filter size tends to zero the residual stresses vanish and we recover (5). It is also possible to obtain a transport equation (and its integral form) for the subgrid kinetic energy $k' = \frac{1}{2} \boldsymbol{u}' \cdot \boldsymbol{u}'$ in which term V also appears [26].

In the discussion below we will consider the standard Smagorinsky LES model, in which the subgrid closure is

$$\tau = \nu^t \nabla^s \bar{\boldsymbol{u}},$$