Comput. Methods Appl. Mech. Engrg. 199 (2010) 853–864

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

An algebraic variational multiscale–multigrid method for large eddy simulation of turbulent flow

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ARTICLE INFO

Article history:
Received 26 September 2008
Received in revised form 22 April 2009
Accepted 27 May 2009
Available online 3 June 2009

Keywords:
Turbulent flow
Large eddy simulation
Variational multiscale method
Scale separation
Algebraic multigrid

ABSTRACT

An algebraic variational multiscale–multigrid method is proposed for large eddy simulation of turbulent flow. Level-transfer operators from plain aggregation algebraic multigrid methods are employed for scale separation. In contrast to earlier approaches based on geometric multigrid methods, this purely algebraic strategy for scale separation obviates any coarse discretization besides the basic one. Operators based on plain aggregation algebraic multigrid provide a projective scale separation, enabling an efficient implementation of the proposed method. The application of the algebraic variational multiscale–multigrid method to turbulent flow in a channel produces results notably closer to reference (direct numerical simulation) results than other state-of-the-art methods both for mean streamwise and root-mean-square velocities. For predicting highly sensitive components of the Reynolds-stress tensor in the context of turbulent recirculating flow in a lid-driven cavity, the algebraic variational multiscale–multigrid method also shows a remarkably good performance in predicting reference results from experiment and direct numerical simulation compared to other methods.

1. Introduction

In the numerical simulation of turbulent flows, adequate computational power to execute a direct numerical simulation (DNS, see, e.g. [1]), which aims at numerically resolving all flow scales, is usually not available. A promising alternative then is large eddy simulation (LES). The strategy of LES consists of resolving the larger flow structures and modeling the effect of the smaller flow structures on the larger structures, see, e.g. [2] for detailed descriptions. It has been learned from Kolmogorov's hypotheses [3] that the smaller scales exhibit a more universal character than the larger scales, which favors a more general validity of a once-developed model for the smaller scales than for the larger scales. As a result, LES appears to be a promising approach in two respects. On the one hand, a coarser discretization, which is substantially coarser than a comparable DNS discretization in the majority of the cases, is sufficient for resolving the larger scales. On the other hand, the universal character of the smaller scales simplifies the modeling process.

A new approach to LES based on the concept of the variational multiscale method (VMM) was introduced in [4]. Two important aspects characterize this approach. First, variational projection separates scale ranges rather than spatial filtering as in a traditional LES. Second, the (direct) influence of the subgrid-scale model, which is introduced to represent the effect of the unresolved scales on the resolved scales, is confined to the finer of the resolved scales. Thus, the coarser scales are solved similar to a DNS (i.e., without any direct influence of the modeling term). Of course, the coarse resolved scales are still indirectly influenced by the subgrid-scale model due to the inherent coupling of all scales. The method was later reinterpreted in the form of a separation of the problem scales into three scale groups in [5,6]. The three-scale separation accounts specifically for "large (or coarse) resolved scales", "small (or fine) resolved scales", and "unresolved (or subgrid) scales". Recently, this "Variational Multiscale LES (VMLES)", as it was called therein, was reviewed in [7]. The reader may also find many references to applications of the VMLES to turbulent flow problems to that date in that publication. Another, more general recent review on LES addressing VMLES, among other things, may be found in [8].

The VMM is a theoretical framework for the separation of scale groups based on the variational formulation of a partial differential equation, relying on a direct sum decomposition of the respective

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doi:10.1016/j.cma.2009.05.017

function spaces. In accordance with this theoretical procedure, it is essential to develop practical implementations by incorporating the VMM framework into a specific numerical method. For the VMLES, the crucial aspect is the implementation of the separation between coarse resolved scales and fine resolved scales. Basically, as proposed in [7], it may be distinguished between "*h*-type" (or grid-based) and "*p*-type" (or polynomial-based) scale separations. One particular way of performing *h*-type scale separation is to make use of procedures well-established in the context of multigrid solvers. Multigrid methods (see, e.g. [9–11]) are among the most efficient iterative algorithms for solving linear systems associated with partial differential equations. The basic idea is to damp errors by utilizing multiple resolutions in an iterative scheme. Oscillatory components are efficiently reduced via a simple smoothing procedure, while smooth components are addressed using an auxiliary lower-resolution version of the discretization (coarse grid). The idea is recursively applied on the next coarser level. Two types of multigrid methods may be distinguished: geometric multigrid (GMG) (see, e.g. [10,11] for elaboration) and algebraic multigrid (AMG).

An initial study on employing GMG operators for VMLES was provided in [12]. The goal of that study was the separation of resolved scales into coarser and finer ones in the context of a finite volume method for LES of turbulent flows. A similar scale separation, inspired by a volume–agglomeration method as used in [13], had been used before in [14], without addressing it from a multigrid perspective, though. Without reference to VMLES, which had not yet been established at that time, GMG approaches to LES had already been proposed in [15] and later in [16], see also the recent review [17]. In contrast to those earlier studies, we will focus on AMG instead of GMG (or volume–agglomeration) methods in the present work. Hence, a particular feature of the method proposed in this study is the opportunity to apply modeling terms either to all scale groups or only to selected scale groups, for instance, only to the finer resolved scales of the problem, as required by VMLES, in a purely algebraic way. This means that neither any coarse discretization besides the basic one (in contrast to all aforementioned GMG-based approaches) nor a challenging geometry-oriented volume–agglomeration procedure (as in [14]) is required.

Regarding the implementation of algebraic multigrid techniques, it may be distinguished between classical AMG, which is sometimes also called "Ruge–Stüben-AMG" [18] as well as its numerous variants and derived techniques, and (smoothed) aggregation- or agglomeration-based AMG (SA-AMG) [19] and derived methods. SA-AMG, as proposed in [19], has been proven to be an optimal method for the solution of elliptic problems which result in symmetric positive definite matrices at the end of the discretization process. In this context, a notion of high-energy and low-energy modes usually replaces the concept of oscillatory and smooth error components mentioned above. However, as soon as hyperbolic terms come into play, resulting in non-symmetric matrix systems, a comparable notion of energy does not exist anymore, and as a consequence, the underlying theory of SA-AMG does no longer hold, see also [20]. An option for such problems is the use of Petrov–Galerkin SA-AMG, see [21,20] for basic considerations and [22] for a recently developed method. Another alternative is plain aggregation AMG (PA-AMG). PA-AMG was, for example, used for solving matrix systems arising from discretizations of the incompressible Navier–Stokes equations in [23]. However, a problem of PA-AMG in the context of solving matrix problems related to such configurations is its sub-optimality when the elliptic part plays a considerable role (e.g., in problems with diffusion-dominated regions). Though conceptually different, PA-AMG is closely related to volume–agglomeration multigrid methods (see, e.g. [13,24,25]), which were preferably developed for finite volume discretizations of hyperbolic problems.

In this study, we will mainly focus on PA-AMG by developing scale-separating operators from the level-transfer operators and incorporating them into our formulation. We will compare numerical results obtained with the proposed method of the present study to results obtained with another method recently developed in the context of the VMM for LES. In [26], a new Residual-Based VMM (RBVMM) was proposed, in which the first of the aforementioned two aspects of VMLES was kept alive (i.e., variational projection for separating scales). Instead of a subgrid-scale model in traditional form, though, an approximate analytical representation of the unresolved scales is used for modeling purposes. The developers of the RBVMM were able to strongly rely on the considerable experience already gained at that time with stabilized finite element methods in fluid mechanics. Localized approximations to Green's function of the respective equation(s) governing a flow problem are essential ingredients of stabilized methods. The reader may find a general overview on stabilized methods in [27]. In fact, the RBVMM may be considered an advanced stabilized method, paying particular attention to the non-linear convective term within the Navier–Stokes equations. As an extension to that method, it was recently proposed in [28] to further take into account the time dependency of the approximate analytical representation of the unresolved scales. This extension was applied to turbulent flow problems in [29], but will not be considered in the present work.

The main contribution of the present work will be an algebraic variational multiscale–multigrid formulation, which brings together VMLES and PA-AMG, two techniques originally proposed in the aforementioned studies, to achieve an accurate and efficient method for the numerical solution of turbulent flows in the sense of LES. Measures for enhancing the efficiency of the method by exploiting the projective property of PA-AMG-based scale-separating operators will be emphasized. According to this, the remainder of the present article is organized as follows. In Section 2, we present the VMLES for the particular case of a multigrid-based scale separation, pointing out all required subgrid-scale modeling steps. The algebraic variational multiscale–multigrid formulation based on PA-AMG is presented in Section 3. The proposed formulation is then tested for two turbulent flow examples, turbulent flow in a channel and turbulent recirculating flow in a lid-driven cavity, in Section 4. Finally, in Section 5, conclusions are drawn from this study.

2. Variational multiscale large eddy simulation using multiple grid levels

2.1. Variational formulation of the incompressible Navier–Stokes equations

The incompressible Navier–Stokes equations in a bounded, connected domain Ω are defined as follows: find a velocity field $\mathbf{u}$ and a pressure field (divided by density) p such that

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - 2\nu \nabla \cdot \varepsilon(\mathbf{u}) + \nabla p = \mathbf{f}, \tag{1}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

for $0 < t \leqslant T$ and $\mathbf{x} \in \Omega$, where $\mathbf{f}$ denotes a given body force, ν the kinematic viscosity of the fluid, and $\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ the rate-of-deformation tensor. At $t = 0$, it is required that $\mathbf{u}(\cdot, 0) = \mathbf{u}_0$ for a prescribed divergence-free initial velocity field $\mathbf{u}_0$. At the boundary $\Gamma \times (0,T] = \partial\Omega \times (0,T]$, Dirichlet and Neumann boundary conditions are prescribed as

$$\mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma_{\mathrm{D}} \times (0,T], \tag{3}$$

$$-p\mathbf{n} + 2\nu\varepsilon(\mathbf{u}) \cdot \mathbf{n} = \mathbf{h} \quad \text{on} \quad \Gamma_{\mathrm{N}} \times (0,T], \tag{4}$$