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Space and time localization for the estimation of distributed parameters in a finite element model

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ABSTRACT

In this paper we study the problem of estimating the possibly non-homogeneous material coefficients inside a physical system, from transient excitations and measurements made in a few points on the boundary. We assume there is available an adequate Finite Element (FEM) model of the system, whose distributed physical parameters must be estimated from the experimental data.

We propose a space–time localization approach that gives a better conditioned estimation problem, without the need of an expensive regularization. Some experimental results obtained on an elastic system with random coefficients are given.

1. Introduction

The Finite Element Method (FEM) [10] is a fundamental numerical tool in modern engineering design. Often, the a-priori knowledge about the system is by itself sufficient to determine the values of the parameters in the FEM model. Nevertheless, there is a growing number of applications in which it is convenient to use a finite element model but there is only a partial knowledge about the system to be modeled and it is necessary to do some system identification from available collections of experimental data. This happens e.g. in the simulation of crash-tests in the automotive engineering [8], where the identification of the system is required for those components (e.g. rubber couplings, solder joints) whose behavior cannot be accurately predicted a-priori. After the identification process, the simulation tool becomes more reliable at reproducing virtual experiments on the system. Here, we turn our attention at applications in which the system identification is used as a diagnostic tool, e.g. in non-destructive control or inspection of mechanical and civil structures. In this case, the final result of the computer application is not the numerical simulation, but the estimate produced by the system identification process, in order to verify the hypothesis about the system integrity or to point out the presence of internal anomalies in the material. In these settings, we assume to know the properties of the material at the surface of the system from physical measurements. Then, we want to estimate the coefficients of the material inside the system, through the propagation sensing of, artificially induced, small elastic vibrations or acoustic waves or thermal heatings, to say a few. This is clearly an inverse problem. In particular, let us consider the situation in which only a small portion of the boundary can be instrumented and, consequently, relatively a few measurement points (sensors) are available. In general, the number and the position of the sensors are fundamental topics to have sufficient information available and a consequent well-conditioned parameter estimation problem, that must be solved numerically. A problem close to the one here considered it is often called in the literature the *inverse scattering* problem [12] and *seismic inversion* is one of the most studied applications. Several algorithms have been proposed in the literature, e.g. travel-time [22], layer-stripping [23], migration [5]. They follow a *signal-based* approach, i.e. that the estimation of the material properties inside the system is performed by monitoring the presence and the properties of backward waves (scattered waves) produced by the presence of interfaces between different materials (e.g. inclusions) inside the system.

In this paper we follow a *model-based* approach, that compares the predictions of the simulation model with the corresponding data coming from the sensors. It does not require to take into account explicitly any scattering paradigm. Actually, here the estimation process starts in the early forward propagation path of the probing signal. The finite element model becomes, in this context, a fundamental mathematical support to the estimation algorithm: it defines precisely the class of distributed parameter models in which to find the model that better describes the experimental data. Moreover, its parametrization takes explicitly into account the physical laws of continuum mechanics. In this way, the identified model is not only a good reproducer of the experimental data, but its coefficients have physical meaning and their values can be

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used to make a physical interpretation of the measured response of the system. To estimate the parameters of a time-dependent FEM model from sampled data, it is convenient to express it in the form of a discrete-time (linear) dynamical system and to use it as the *reference model* in a Kalman filter/predictor (see e.g. [9,19]). The Prediction Error Method (PEM) [13,16] is often used in the system identification practice. It computes the values of the model parameters which minimize a quadratic cost functional of the prediction error, produced e.g. by the Kalman filter, that depends nonlinearly from the model parameters. An alternative approach is that of using the Extended Kalman Filter (EKF) to estimate the model parameters directly as state-variables of the extended filter [4,6]. Another approach is to use the so-called *subspace* methods (see e.g. [14] for a recent survey), in which the state is determined first by projection, with techniques based on the Singular Value Decomposition (SVD), and then the model parameters are estimated with a least squares technique. Here we have chosen the PEM approach because it is more suited to estimate physical parameters, which enter nonlinearly in the system matrices. In this paper we will show that, in case of models with distributed parameters, it is preferable to estimate sequentially the parameters in small subregions, instead of performing a global estimate. For this purpose, we present a *space–time localized* PEM method.

The paper is organized as follows. In Section 2 it is presented the continuous model problem and its discretization. Section 3 presents the parameter estimation issues and the proposed algorithm (see also Appendix A). In Section 4 are shown some numerical experiments.

2. Model problem

Let us consider the well-known wave equation that, in a one-, two- and three-dimensional domain, describes e.g. the vertical oscillations of a string, of a membrane or the propagation of an acoustic wave within a solid. In particular, we will consider a 1D model with randomly varying material coefficients:

$$\rho(x)\frac{\partial^2 d}{\partial t^2} - \nabla \cdot k(x)\nabla d = f, \tag{1}$$

where $d = d(x,t)$ is the instant displacement at the time t of a material point initially located at the coordinate $x \in \Omega \subset \Re^q, q = 1,2,3, f = f(x,t)$ is the forcing term (an external force), $\rho(x)$ is the material density and $k(x)$ is the material stiffness. Moreover, we suppose to have: forcing terms which are localized in a small region Ω_f of the space domain; an a-priori knowledge (e.g. by direct experimental measurement) about the values of the material coefficients in Ω_f; sensors located in positions that are close to Ω_f. The problem here considered is to estimate the material coefficients outside Ω_f by means of experimental data given by the sensors. It represents typical situations arising e.g. in non-destructive testing or in soil exploration. From the point of view of system identification, this is a situation in which it is not possible to choose the number and the location of the sensors, an issue that usually would improve the conditioning of the parameter estimation problem, see e.g. [8].

Let us apply the Finite Element Method (see e.g. [10]) to discretize in space our model problem. As is well-known, we obtain the following system of ordinary differential equations:

$$M\ddot{d}_h(t) + Kd_h(t) = f_h(t), \tag{2}$$

where M and K are called, respectively, the mass and stiffness matrices. The vector $f_h(t)$ is the discretized forcing term. Let it be n_f and e_f, respectively, the mesh nodes and elements contained in the region Ω_f. For simplicity, and with great generality, we assume that the material coefficients $\rho(x)$ and $k(x)$ are randomly set in all Ω and can be well approximated by a piecewise constant function defined on the finite element mesh. Moreover, since in our model problem they are known only in the elements e_f and unknown in the other ones, we assume that the values of the material coefficients in the elements outside Ω_f are the unknown model parameters, that we represent with the vector θ. Therefore, the spatial resolution of the parameter estimates depends on the local size of the finite element mesh. The optimal parametrization can be conveniently built using an adaptive finite element discretization [15,3,1]. Moreover, in our settings the parameters of the model are the physical coefficients of the material. In this way, the value of the estimated parameters turns out to be independent of the geometric shape of the elements and therefore the use of a dynamic mesh, in particular an adaptive mesh [21], do not introduce perturbations to the convergence of the estimates.

Let us define a state vector $x(t) = \begin{bmatrix} \dot{d}_h(t) \\ d_h(t) \end{bmatrix}$ and an input vector $u(t) = f_h(t)$, to obtain:

$$\dot{x}(t) = Ax(t) + Bu(t), \quad A = \begin{bmatrix} 0 & -M^{-1}\cdot K \\ I & 0 \end{bmatrix}, \quad B = \begin{bmatrix} M^{-1} \\ 0 \end{bmatrix}. \tag{3}$$

Various methods may be used to discretize (3) in time (see e.g. [10,20]). For example, if we apply the implicit Euler scheme we obtain the following recurrent formula (τ is the time-step, and n is the discrete-time index of the instant $t_n = n\tau$):

$$(I - \tau A)x_{n+1} = x_n + \tau B u_{n+1}. \tag{4}$$

Now, let us assume that the measurement vector $y_n = y(t_n)$ is a linear combination of the state variables and consider two additive stochastic vector processes $\{v_n\}$ (*model noise*) and $\{w_n\}$ (*measurement noise*), which we assume zero-mean and uncorrelated. We obtain a discrete-time, state-space, linear, stochastic dynamical system:

$$x_{n+1} = Ax_n + Bu_n + v_n, \tag{5}$$
$$y_n = Cx_n + Du_n + w_n. \tag{6}$$

The model and measurement noise vectors are characterized by their covariance matrices, respectively Q and R. The covariance matrix Q must be tuned to describe the "statistical power" of the modeling approximation error, while the covariance R of the measurement noise can be deduced from the precision of the measurement devices. To the model (5) and (6) it is now possible to apply the discrete-time Kalman filter [9,19]. It produces an optimal prediction of the outputs $\hat{y}_{n,\theta} = C\hat{x}_{n|n-1}$ having observed the experimental data $\tilde{y}_{n-1}$, i.e. up to the time $n-1$, and having consequently predicted the state vector $\hat{x}_{n|n-1}$, where the index "$n|n-1$" means "at time n given the state (estimate) at time $n-1$". The index θ means that $\hat{y}_{n,\theta}$ is obtained from a reference model with parameter values θ, which are different, in general, from the true ones. The *prediction error* has the following expression:

$$\epsilon_{n,\theta} = \tilde{y}_n - \hat{y}_{n,\theta}. \tag{7}$$

If (5) and (6) is a good model of the system, there is no systematic modeling error and with a well-tuned filter the prediction error has no deterministic components, i.e. it is white noise.

3. Model parameters estimation

In this paper, we suppose that the measurement errors are sensibly smaller than the estimation errors. In fact, in our case the estimation of the state x_n is performed with a reference model whose parameters are only partially known, i.e. there is in general a relatively high level of modeling error.