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n-Widths, sup–infs, and optimality ratios for the *k*-version of the isogeometric finite element method

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A B S T R A C T

We begin the mathematical study of the *k*-method utilizing the theory of Kolmogorov *n*-widths. The *k*-method is a finite element technique where spline basis functions of higher-order continuity are employed. It is a fundamental feature of the new field of isogeometric analysis. In previous works, it has been shown that using the *k*-method has many advantages over the classical finite element method in application areas such as structural dynamics, wave propagation, and turbulence.

The Kolmogorov *n*-width and sup–inf were introduced as tools to assess the effectiveness of approximating functions. In this paper, we investigate the approximation properties of the *k*-method with these tools. Following a review of theoretical results, we conduct a numerical study in which we compute the *n*-width and sup–inf for a number of one-dimensional cases. This study sheds further light on the approximation properties of the *k*-method. We finish this paper with a comparison study of the *k*-method and the classical finite element method and an analysis of the robustness of polynomial approximation.

1. Introduction

In this paper, we present a theoretical and computational framework that allows one to examine approximation properties of a prescribed discretization. The framework presented in this paper is based on the theory of Kolmogorov *n*-widths. This theory defines and gives a characterization of optimal *n*-dimensional spaces for approximating function classes and their associated errors. *n*-Widths are a well-explored subject in approximation theory, but they are not as familiar to the finite element and computational mechanics communities.

A practically useful concept that emerges from the theory of *n*-widths is the sup–inf. Sup–infs quantify the error induced by a particular discretization in approximating a given class of functions. In the context of Hilbert spaces, sup–infs can be directly computed by way of the solution of a variational eigenproblem. As error is exactly quantified, sup–infs can distinguish between two methods of the same approximation order. Although such distinctions are rarely made in classical approximation theory of finite elements, we feel that such comparisons are necessary, primarily due to the advent of new computational technologies. For example, C^0- and C^1-continuous quadratic finite elements deliver the same asymptotic convergence rate, but the size of the approximation errors for the two classes of functions will be different. By comparing the sup–inf to the *n*-width, we are able to assess the performance of a given approximation space with respect to the optimal discretization.

Recently Babuska et al. [4] made use of *n*-widths and sup–infs to assess approximation properties of functions employed in generalized finite element methods. In this paper, we apply this framework to the study of one-dimensional spline spaces of variable order and continuity. Particular instances of these spaces include classical C^0-continuous finite elements as well as global polynomials. The emphasis of our study is spline functions of maximal continuity. Such functions are the basis of the *k*-version of the isogeometric finite element method. The concept of isogeometric analysis was first introduced in [21]. The developments of [21] aimed at unifying geometrical modeling and analysis for engineering applications. Although the development of isogeometric analysis was driven by the need of a tighter link between computer-aided design and computer-aided analysis, using spline functions in analysis has proven to be beneficial from the standpoint of solution accuracy. Recent results in structural vibrations [13,14], wave propagation [22], and turbulent flow [1,8] indicate that on a per degree-of-freedom basis, discretizations of higher-continuity are superior to their C^0-continuous counterparts.

In Section 2, we give an overview of spline functions, their construction, and their properties and define four discrete function spaces used for the analysis in this paper. In Section 3, we outline particular function classes of interest that arise in the study of partial differential equations that we wish to approximate with spline

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functions. These classes of interest include standard Sobolev spaces, Sobolev spaces with periodic boundary conditions, and Jacobi-weighted Sobolev spaces. In Section 4, we introduce terminology and concepts and state classical results of the theory of n-widths and sup–infs. In this section, we also present known theoretical results on the optimality of splines for function classes of interest. In Section 5, we outline a computational framework for the evaluation of n-widths and sup–infs and apply this framework to a number of one-dimensional cases. In Section 6, we draw conclusions based on our studies.

2. Splines

This section gives a very brief overview of univariate B-splines and periodic splines. B-splines were first introduced by Schoenberg in 1946 [28] in the attempt of developing piecewise polynomials with prescribed smoothness properties. In his 1972 paper, de Boor [15] introduced a simple and stable recursion formula for evaluating them, and since then, B-splines have been a standard in the numerical analysis and computer-aided geometric design communities. For an overview of splines, their properties, and robust algorithms for evaluating their values and derivatives, see de Boor [16] and Schumaker [29]. For an introductory text on non-uniform rational B-splines (NURBS), see Rogers [27], while more detailed treatments are given in the books of Piegl and Tiller [25] and Cohen et al. [12]. For the application of splines to finite element analysis, see Höllig [20] and Hughes et al. [21].

A *B-spline* basis is comprised of piecewise polynomials joined with prescribed continuity. In order to define a B-spline basis of polynomial order p in one dimension one needs the notion of a *knot vector*. A knot vector in one dimension is a set of coordinates in the parametric space, written as $\Xi = \{\xi_1, \xi_2, \ldots, \xi_{n+p+1}\}$, where i is the knot index, $i = 1, 2, \ldots, n+p+1$, $\xi_i \in \mathbb{R}$ is a knot, $\xi_1 \leqslant \xi_2 \leqslant \cdots \leqslant \xi_{n+p+1}$, and n is the total number of basis functions.

Given Ξ and p, B-spline basis functions are constructed recursively starting with piecewise constants ($p = 0$):

$$B_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i \leqslant \xi < \xi_{i+1}, \\ 0 & \text{otherwise}. \end{cases} \quad (1)$$

For $p = 1, 2, 3, \ldots$ they are defined by

$$B_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} B_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} B_{i+1,p-1}(\xi). \quad (2)$$

When $\xi_{i+p} - \xi_i = 0$, $\frac{\xi - \xi_i}{\xi_{i+p} - \xi_i}$ is taken to be zero, and similarly, when $\xi_{i+p+1} - \xi_{i+1} = 0$, $\frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}}$ is taken to be zero.

We define

$$S(n, p, \Xi) = \text{span}\{B_{1,p}(\xi), B_{2,p}(\xi), \ldots, B_{n,p}\} \quad (3)$$

to be a B-spline space of dimension n with degree p and built using knot vector Ξ. B-spline basis functions form a partition of unity, each one is compactly supported on the interval $[\xi_i, \xi_{i+p+1}]$, and they are point-wise non-negative. These properties are important and make these functions attractive for use in analysis.

The first and last knots are called *end knots*, and the other knots are called *interior knots*. Note that knots may be repeated. A knot vector is said to be *open* if its end knots have multiplicity $p + 1$. Basis functions formed from an open knot vector are interpolatory at end knots of the parametric interval but they are not, in general, interpolatory at interior knots. Basis functions of order p have $p - 1$ continuous derivatives at non-repeated knots. If a knot has multiplicity k, then the number of continuous derivatives decreases by $k - 1$. When the multiplicity of a knot is exactly p, the basis function is interpolatory and only C^0-continuous at that knot.

Periodic splines are constructed from B-splines subject to periodic boundary conditions. If one desires a periodic spline space that is C^{s-1} at the end knots, one must directly enforce this constraint onto the associated B-spline space by restricting the first $s - 1$ derivatives at the end knots to be equal.

The above constructions (1,2) encompass a large class of functions. All the finite-dimensional spaces considered in this paper may be expressed using particular instantiations of the knot vector Ξ. In particular, we define the following discrete spline spaces:

- $K(n, k, a, b)$ is the B-spline space of dimension n and degree k corresponding to an open knot vector Ξ with end knots located at a and b (with $b > a$) and with *equispaced* and *non-repeated* interior knots. Because the interior knots are distinct, the functions in this space attain maximal continuity at interior knots. That is, $K(n, k, a, b) \subset C^{k-1}(a, b)$.
- $K_{per}(n, k, a, b)$ is the periodic spline space of dimension n and degree k corresponding to the open knot vector Ξ with end knots located at a and b and with *equispaced* and *non-repeated* interior knots, subject to periodic boundary conditions of maximal continuity. Namely, if $u \in K_{per}(n, k, a, b)$, then

$$D^{(i)}u(a) = D^{(i)}u(b), \quad \forall i = 1, 2, \ldots, k - 1. \quad (4)$$

  This space is known as the space of *uniform periodic splines*. Maximal order of continuity at all knots is attained in this case.
- $P(n, p, a, b)$ is the B-spline space of dimension n and degree p corresponding to the open knot vector Ξ with end knots located at a and b, $b > a$, and with *equispaced* interior knots, each of which is repeated $p - 1$ times. This is a space of *standard* finite element functions of degree p that are C^0-continuous at knots.
- $\mathscr{P}(n, a, b)$ is the B-spline space of dimension n and degree $n - 1$ corresponding to the open knot vector Ξ with end knots located at a and b, $b > a$, and with no interior knots. This is a space of global polynomials of degree $n - 1$.

In this paper, we define the *k-version of the isogeometric finite element method* or, in short, the *k-method* as the analysis method exploiting full continuity of the basis functions across distinct knots (and hence results from knot vectors with non-repeated interior knots). Alternatively, we define the *classical finite element method* as the analysis method where only C^0-continuity is enforced across interior knots. With these definitions, we see that the spaces $K(n, k, a, b)$ and $K_{per}(n, k, a, b)$ correspond to the k-method while the space $P(n, p, a, b)$ corresponds to the classical finite element method of degree p. The space $\mathscr{P}(n, a, b)$ corresponds to the *spectral method* or the *p-version of the finite element method* and is a special case of both the classical finite element and k-methods, the case where no interior knots exist. Examples of these spaces are illustrated in Fig. 1.

3. Function spaces

In this section, we introduce a number of function spaces which arise in the study of elliptic partial differential equations. Solutions to such equations often live in these function spaces, depending of course on the regularity of the underlying problem and associated boundary conditions.

3.1. Sobolev spaces

Let $\Omega \subset \mathbb{R}$ be an open domain. For an integer $m \geqslant 0$, we use the notation

$$H^m(\Omega) = \left\{ u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \text{ for all } \alpha = 0, \ldots, m \right\}. \quad (5)$$

The Sobolev space $H^m(\Omega)$ is a Hilbert space with inner product