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# A discontinuous Galerkin formulation of a model of gradient plasticity at finite strains ☆

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ABSTRACT

The work presented here constitutes an extension to the finite-strain regime of a discontinuous Galerkin based, strain gradient plasticity formulation presented in Djoko et al. [J.K. Djoko, F. Ebobisse, A.T. McBride, B.D. Reddy, A discontinuous Galerkin formulation for classical and gradient plasticity – Part 1: formulation and analysis, Comput. Methods Appl. Mech. Engrg. 196 (2007) 3881–3897; J.K. Djoko, F. Ebobisse, A.T. McBride, B.D. Reddy, A discontinuous Galerkin formulation for classical and gradient plasticity – Part 2: algorithms and numerical analysis, Comput. Methods Appl. Mech. Engrg. 197 (2007) 1–21]. The focus here is on algorithmic and computational aspects of the formulation at finite strains. The adoption of a logarithmic hyperelastic–plastic formulation preserves the essential features of the infinitesimal formulation. This key ingredient allows the predictor–corrector solution algorithms developed for the infinitesimal gradient formulation to be extended readily to the finite-strain regime. The use of low-order elements is essential to contain the computational expense of the formulation but these elements are prone to locking. The method of enhanced assumed strains for geometrically nonlinear problems is utilised to circumvent this limitation. The form of the consistent tangent modulus is derived for the case of gradient plasticity. Two numerical examples are presented to illustrate aspects of the approximation scheme and the algorithm, as well as features of the model of gradient plasticity.



## 1. Introduction

The inability of the well-developed classical theories of plasticity to capture scale-dependent behaviour is one of the primary motivations for the development of a range of strain gradient plasticity models that attempt to represent the underlying mesoscale phenomena within a continuum framework, see, for example, [1]. A further motivation concerns the inability of the classical models to describe softening media. In the early works of Dillon and Kratochvil [2], Aifantis [3,4] and Coleman and Hodgdon [5], the von Mises yield criterion is augmented by a term involving the Laplacian of the equivalent plastic strain, and possibly further higher-order terms. These theories incorporate, in a natural way, a physical length scale, and thereby allow phenomena such as shear banding to be represented meaningfully. The relation of these theories of gradient plasticity to the underlying interpretation of plastic deformation arising due to the flow of dislocations in the crystal lattice structure was established by Aifantis [3,4].

The nonstandard higher-order contributions arising in gradient plasticity formulations render the conventional framework of classical finite elements inappropriate. Various approaches have hitherto been used in the numerical treatment of problems in gradient plasticity based upon a similar model to the one considered here. De Borst and Mülhaus [6] derived a weak form of the gradient plasticity formulation proposed by Mülhaus and Aifantis [7] as well as the resulting finite element framework. The use of $C^1$ finite element formulations for the interpolation of the hardening parameter has been documented in [6,8–10]. Related work, using a conforming approximation, is that of Liebe and Steinmann [11]. De Borst et al. [12] extended their earlier work to include gradient damage within a gradient plasticity formulation. Other contributions concerned with gradient damage include the investigation by Wells et al. [13] while Garikipati [14] has explored a variational multiscale approach to a model of gradient plasticity proposed by Fleck and Hutchinson [1].

The work presented here constitutes an extension to the finite-strain regime of a discontinuous Galerkin based, strain gradient plasticity formulation documented for the infinitesimal theory in [15,16]. Various other researchers have also considered the extension of the gradient plasticity model in [7] to the finite-strain regime (see, for example, [17–20]). An evaluation of the ability of several higher-order plasticity theories to predict size effects and

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localisation was presented by Engelen et al. [21]. Other key contributions to the numerical simulation of problems of gradient plasticity include those presented in [22–27,20,28], amongst others.

As detailed in [15,16], the discontinuous Galerkin finite element method allows the higher-order contributions arising in the gradient formulation to be treated in an elegant and effective manner. In discontinuous Galerkin methods, interelement continuity of the approximation field is relaxed in a framework in which the discrete problem remains consistent.

Discontinuous Galerkin methods were developed in the 1970s and 1980s [29,30], but it is only in recent years that they have been exploited in a wide range of problems. The collection [31] provides an excellent overview of the key approaches for elliptic and hyperbolic problems. Within the context of linear elasticity there have been important contributions by Rivière and Wheeler [32] and Wihler [33], the latter considering the case of nonconvex domains and vanishing compressibility. Ten Eyck and Lew [34] demonstrated the effectiveness of the discontinuous Galerkin formulation in circumventing locking-related problems arising due to vanishing compressibility within the context of nonlinear elasticity. A key contribution of their work was to show that the discontinuous Galerkin formulation produced results of similar accuracy to those obtained using a conforming approximation with a comparable, and often lower, computational cost. The effective treatment of the incompressibility constraint is of significant importance in many models of plasticity in which plastic deformation is assumed incompressible.

A discontinuous Galerkin method has recently been developed for strain gradient dependent damage models [13,35], while the work by Engel et al. [36] treats continuous/discontinuous Galerkin methods for fourth-order problems by reducing the classical requirement of $C^1$ continuity of the unknown variable to one of continuity. Discontinuous Galerkin in time approximations for classical plasticity have been investigated by Alberty and Carstensen [37]. More recently, the discontinuous Galerkin method has been applied to problems in nonlinear elasticity by ten Eyck and Lew [34] and Noels and Radovitzky [38].

The extension of our previous work to the finite-strain regime is facilitated by the adoption of a logarithmic hyperelastic–plastic model that preserves the essential ingredients of the return mapping algorithms of the infinitesimal theory. The model was developed for classical plasticity by Simo [39]. The extension of the classical small-strain plasticity theory to finite strains using logarithmic strain measures has a considerable history [40–47]. The simplicity of this model of plasticity has been exploited by Geers [48] as the basis for a nonlocal implicit gradient plasticity formulation at finite strains.

The assumption of incompressible plastic deformation in the von Mises yield criterion renders a finite element solution using low-order elements susceptible to volumetric locking. Low-order elements are advantageous however as they reduce the computational expense of the formulation and are more robust than high-order elements for large-deformation problems. The method of enhanced assumed strains for geometrically nonlinear problems, originally developed by Simo and Armero [49] and extended in subsequent works [50–52], is utilised to provide a locking-free response for low-order elements.

This work focuses on algorithmic and computational aspects of the model of gradient plasticity considered at finite strains. In Section 2 we review the relations governing an elastoplastic body. The method of enhanced assumed strains is outlined. Relevant terminology pertaining to the discontinuous Galerkin finite element method is then presented in Section 3. This provides the background for the discontinuous Galerkin formulation of the nonlocal consistency condition arising in the gradient problem. The numerical solution of the gradient plasticity problem is realised by means of a predictor–corrector algorithm, as discussed in Section 4. In addition to deriving the algorithmic consistent tangent modulus, full details of the implementation of the algorithm are given. Two example problems are presented in Section 5 to illustrate the performance and key features of the algorithm. The work concludes with a summary and a review of possible extensions.

## 2. The governing equations for the problem

We denote by $\Omega \in \mathbb{R}^2$ the reference placement of a continuum with material points denoted $\boldsymbol{X}$ as depicted in Fig. 1. The time domain under consideration is the interval $[0, T]$. The boundary of $\Omega$ is denoted by $\partial\Omega$ with outward normal $\overline{\boldsymbol{N}}$. Dirichlet and Neumann boundary conditions for the displacement $\boldsymbol{u}$ and the traction $\boldsymbol{T}$ are prescribed on $\Gamma_\varphi$ and $\Gamma_T$, respectively, in addition $\Gamma_\varphi \cap \Gamma_T = \varnothing$ and $\overline{\Gamma_\varphi \cup \Gamma_T} = \overline{\partial\Omega}$. The nominal prescribed traction on $\Gamma_T$ is denoted by $\boldsymbol{T} = \boldsymbol{P}\overline{\boldsymbol{N}}$, where $\boldsymbol{P}$ is the first Piola–Kirchhoff stress tensor. The current placement of the continuum body resulting from a deformation $\boldsymbol{\varphi}$ is denoted by $\mathcal{S} = \boldsymbol{\varphi}(\Omega)$. The displacement of a material point relative to the reference configuration is denoted by $\boldsymbol{u}(\boldsymbol{X}, t) = \boldsymbol{\varphi}(\boldsymbol{X}, t) - \boldsymbol{X}$.

Let $\mathscr{T}_h^0 = \{K^0\}$ be a shape-regular subdivision of the reference domain $\Omega$ where $K^0$ are, here, conforming quadrilateral subdomains (finite elements). We denote by $h_K^0 = \mathrm{diam}[K^0]$ a measure of the element size and by $h = \max\{h_k^0, K^0 \in \mathscr{T}_h^0\}$ a measure of the maximum element size in the discretisation.

Following standard finite element procedure we approximate the displacement field $\boldsymbol{u}$ with a trial function $\boldsymbol{u}_h \in V^h$, where $V^h$ is a finite-dimensional subspace of $V = H^1(\Omega)^2$. Over each element we associate a finite-dimensional function space $X$ with basis functions $N_\varphi^A$ $(A = 1, \ldots, n^{\mathrm{e}}_{\mathrm{node}})$ defined on the reference element $\hat{K} \in \square = [-1, -1] \times [1, 1]$. The variable $n^{\mathrm{e}}_{\mathrm{node}}$ denotes the number of nodes per element.

The interpolation of the reference domain and the displacement field over a typical element follow as per conventional Galerkin finite element procedure based upon the isoparametric concept; that is,

$$\boldsymbol{X} \approx \boldsymbol{X}_h = \sum_{A=1}^{n^{\mathrm{e}}_{\mathrm{node}}} N_\varphi^A(\xi)\boldsymbol{X}_A \quad \text{and} \quad \boldsymbol{u}_h = \sum_{A=1}^{n^{\mathrm{e}}_{\mathrm{node}}} N_\varphi^A(\xi)\boldsymbol{u}_A,$$

where $\boldsymbol{X}_A$ and $\boldsymbol{u}_A$ are the reference coordinates and the displacement of node $A$, respectively, and $\xi = (\xi, \eta) \in \square$ are coordinates in the reference element.

The focus of the work presented here is on a formulation of gradient plasticity that utilises the discontinuous Galerkin approach. For the sake of convenience, the displacement field is approximated using conforming bilinear elements. The analysis of the small-strain problem of gradient plasticity using the discontinuous Galerkin method for the approximation of both the displacement and plastic strain fields has been presented by the authors in [15]. Discontinuous Galerkin approximations for nonlinear elasticity have also been the subject of recent investigations by ten Eyck and Lew [34] and Noels and Radovitzky [38]. Bilinear elements were chosen here to decrease the computational expense of the formulation and for their superior robustness over higher-order elements for large-deformation processes.

The deformation gradient is interpolated across an element from the current nodal positions $\boldsymbol{x}_A := \boldsymbol{X}_A + \boldsymbol{u}_A$ as follows:

$$\mathrm{GRAD}_X[\boldsymbol{\varphi}_h] = \sum_{A=1}^{n^{\mathrm{e}}_{\mathrm{node}}} \boldsymbol{x}_A(t) \otimes \mathrm{GRAD}_X[N_\varphi^A], \tag{1}$$

where $\mathrm{GRAD}_X[(\cdot)] := \partial(\cdot)/\partial\boldsymbol{X}$ is the gradient operator with respect to the reference configuration. The time-independent derivatives of