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A variational principle for dynamic contact with large deformation

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Professor J. Tinsley Oden, on occasion of his 70th birthday.

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ABSTRACT

Dynamic contact between a deformable body with either large or small deformation and a rigid obstacle can be modeled with Signorini contact condition stated in terms of the normal component of the displacement – as long as the relative positions are clearly defined. The geometrical relation, however, is unpredictable when large motion or large deformation and complex surface geometry are involved. The use of normal velocity is more suitable. A variational inequality is proposed for the dynamic contact problem. The penalty method is introduced and implemented in explicit finite element software. Numerical examples are presented to demonstrate the robustness of the algorithm.

1. Introduction

During large deformation such as in a vehicle crash, different parts of the structure can come in contact with each other. Even different parts of the same component can be in contact. Thus, in numerical simulation, the contact algorithm is essential in preventing structural penetration so that the analysis can represent the true physical event. Contact problem has been one of the most actively researched fields in applied mechanics, numerical methods, and applied mathematics. It is also a very active field related to the development of software and computer architecture. Major developments in contact mechanics have been illustrated in several well-written books concentrating on contact problems with references being quoted extensively. For example, see [4,7,12,21] for mathematical theories; [10,25,28] for the mechanics aspects and numerical approaches. In addition, the survey articles are the invaluable resources, cf. [2,24,29].

Consider a material body occupying domain Ω in a three dimensional space in contact with a fixed rigid obstacle. This is a type of Signorini's problems, depicted in Fig. 1. The contact is by its portion of boundary Γ_c. Denote by $S(\boldsymbol{x}) = 0$ for the surface of obstacle and assume that S is a smooth function. Let $\boldsymbol{x}_c \in \Gamma_c$ be in contact with the obstacle. $\boldsymbol{x}_c$ satisfies the geometrical condition

$$S(\boldsymbol{x}_c) = 0. \tag{1}$$

Due to the obstacle, the motion of point $\boldsymbol{x}_c$ is restricted unilaterally without penetrating the surface $S(\boldsymbol{x}) = 0$. The non-penetration condition can be described as a kinematic constraint condition, which is usually represented by the normal component of displacement and stress, cf. [12],

$$u_N = \boldsymbol{u} \cdot \boldsymbol{N} \leqslant 0, \qquad \sigma_N = \sigma_{ij} N_i N_j \leqslant 0, \qquad \sigma_N u_N = 0, \tag{2}$$

where $\boldsymbol{N}$ is the unit outer normal vector at $\boldsymbol{x}_c$ on Γ_c. σ_N, associated with stress σ_{ij}, is equal to the normal component of the surface traction. The third equation of (2) is the Kuhn–Tucker complementary condition.

Note that $\boldsymbol{u}(t, \boldsymbol{X})$ is a measure of the displacement happened from time = 0 to t. In general, condition (2) about $\boldsymbol{u}(t, \boldsymbol{X})$ does not necessarily represent the impenetrability after contact occurs at time t. In cases of large deformation, depicted in Fig. 2, the displacement of large deformation can be irrelevant to the normal direction at the points in the surrounding area. The displacement is not really constrained by (2). The normal also changes direction due to large rotation accompanied with large deformation. The normal will lose its meaning as reference for measurement of large deformation by long time duration and the displacement condition becomes inappropriate. We adopt the constraint condition in a rival form with velocity

$$\dot{u}_N = \dot{\boldsymbol{u}} \cdot \boldsymbol{N} \leqslant 0, \qquad \sigma_N = \sigma_{ij} N_i N_j \leqslant 0, \qquad \sigma_N \dot{u}_N = 0. \tag{3}$$

The velocity form of contact condition (3) is usually considered as the first order approximation of the displacement form (2). Ref.

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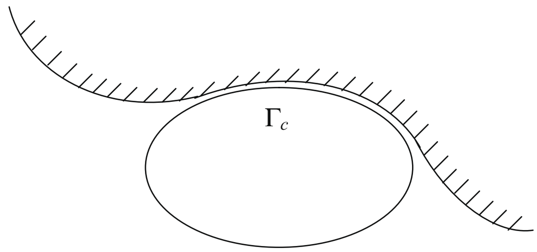

Fig. 1. Contact with an obstacle.

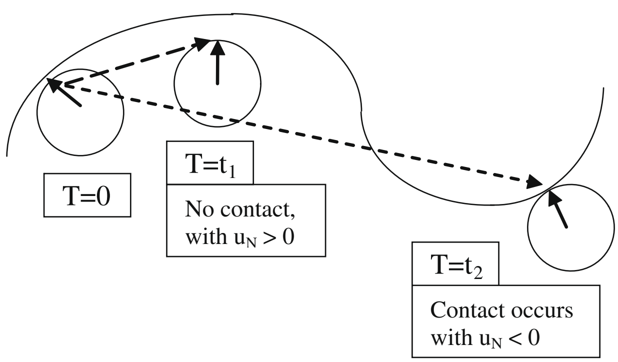

Fig. 2. Dynamic contact with large deformation.

[23] provided reasoning from mechanics point of view. See also [25] for more discussions.

We adopt the Coulomb's friction model, described mathematically, e.g., in [3],

$$|\boldsymbol{F}_T| \leqslant \mu F_N,$$
$$|\boldsymbol{F}_T| < \mu F_N \Rightarrow \dot{\boldsymbol{u}}_T = 0, \tag{4}$$
$$|\boldsymbol{F}_T| = \mu F_N \Rightarrow \exists\lambda > 0, \quad \dot{\boldsymbol{u}}_T = -\lambda \boldsymbol{F}_T,$$

where F_N is the normal component of the surface traction, only compression as positive is considered. $\boldsymbol{F}_T$ is the tangential force vector and μ is the friction coefficient. $\boldsymbol{u}_T$ represents the tangential displacement vector. The second relation of (4) is obvious, reflecting the "sticking" situation with static friction. The third relation bares the simple explanation that the friction force is in the opposite direction of motion.

2. Variational principle for the dynamic contact problem

Assume that part of the boundary, Γ_u and Γ_s, are subjected to prescribed displacement and traction respectively. Γ_u and Γ_s can never be in contact with the obstacle. A portion of the rest of the boundary, on which no displacement nor force is prescribed, can be in contact or not in contact with the obstacle, and that can happen at some times or at all times. If a portion of such boundary is not in contact at time t, we consider that zero traction is applied to it. This kind of boundary may change in time. It is included in Γ_c as contact boundary. The whole boundary Γ is now decomposed into three disjoint parts.

To consider friction in the dynamic system, we need to examine the tangential component σ_{Ti}, whereas the condition for the normal component is still valid. We have the dynamic frictional contact problem described as:

Problem D.

$$\rho\ddot{u}_i - \sigma_{ij,j} = f_i(t, \boldsymbol{x}) \quad \text{in } \Omega, i,j = 1,2,3, \tag{5a}$$
$$u_i(0,\boldsymbol{x}) = U_i^0(\boldsymbol{x}); \quad \dot{u}_i(0,\boldsymbol{x}) = U_i^1(\boldsymbol{x}) \quad \text{in } \Omega, \tag{5b}$$
$$u_i = U_i(t) \quad \text{on } \Gamma_u, \tag{5c}$$
$$\sigma_{ij} n_j = g_i(t, \boldsymbol{x}) \quad \text{on } \Gamma_s, \tag{5d}$$

$$\begin{cases} \dot{u}_N \leqslant 0; \quad \sigma_N \leqslant 0; \quad \sigma_N \dot{u}_N = 0 \\ |\boldsymbol{\sigma}_T| \leqslant \mu|\sigma_N|; \quad \text{if } |\boldsymbol{\sigma}_T| < \mu|\sigma_N| \quad \text{then } \dot{u}_{Ti} = 0 \\ \qquad\qquad \text{if } |\boldsymbol{\sigma}_T| = \mu|\sigma_N| \quad \text{then } \exists\lambda > 0, \quad \dot{u}_{Ti} = -\lambda\sigma_{Ti} \end{cases} \quad \text{on } \Gamma_c, \tag{5e}$$

$$\dot{\sigma}_{ij} = \dot{\sigma}_{ij}(E, \nu, E_t, x_i, u_i, \dot{\varepsilon}_{ij}, \ldots), \tag{5f}$$
$$\dot{\varepsilon}_{ij} = (\dot{u}_{i,j} + \dot{u}_{j,i})/2. \tag{5g}$$

Here, u_{Ti} and σ_{Ti} in (5e) are the tangential displacement and tangential traction respectively. The constitutive relation (5f) in rate form is employed for the nonlinear analysis. E and ν are the Young's modulus and Poisson ratio of elasticity. E_t is the tangent modulus for nonlinear material. Using the test functions v_i, in the same function set of velocity $\dot{u}_i$, with $(v_i - \dot{u}_i)|_{\Gamma_u} = 0$, the variational principle can be written as

$$\int_\Omega (\rho\ddot{u}_i(v_i - \dot{u}_i) + \sigma_{ij}(v_{i,j} - \dot{u}_{i,j}))d\Omega - \int_{\Gamma_C} \sigma_{ij} N_j (v_i - \dot{u}_i) d\Gamma$$
$$= \int_\Omega f_i(v_i - \dot{u}_i) d\Omega + \int_{\Gamma_S} g_i(v_i - \dot{u}_i) d\Gamma. \tag{6}$$

Note that on the portion of boundary Γ_c without contact, $\sigma_N = 0$, but $\dot{u}_N \leqslant 0$ is not necessary. This portion of Γ_c has no contribution to (6). For simplicity, here we consider Γ_c as the portion of boundary really in contact with the obstacle.

With velocity and stress decomposed into normal and tangential components, we have

$$\int_{\Gamma_C} \sigma_{ij} N_j (v_i - \dot{u}_i) d\Gamma = \int_{\Gamma_C} (\sigma_N(v_N - \dot{u}_N) + \sigma_{Ti}(v_{Ti} - \dot{u}_{Ti})) d\Gamma. \tag{7}$$

Ref. [15] used this approach to treat the non-homogeneous boundary value problems. Ref. [3] dealt with the contact problems with non-homogeneous boundary conditions. The test function v_i satisfies the first condition of (3) and σ_{ij} satisfies the second one. It follows that

$$\sigma_N v_N \geqslant 0. \tag{8}$$

Note that the Kuhn–Tucker condition is satisfied only by the true solution, not necessarily by the arbitrary test function. Hence we obtain an inequality

$$\sigma_N(v_N - \dot{u}_N) \geqslant 0. \tag{9}$$

With (7) and (9), Eq. (6) leads to

$$\int_\Omega (\rho\ddot{u}_i(v_i - \dot{u}_i) + \sigma_{ij}(v_{i,j} - \dot{u}_{i,j}))d\Omega - \int_{\Gamma_c} \sigma_{Ti}(v_{Ti} - \dot{u}_{Ti}) d\Gamma$$
$$\geqslant \int_\Omega f_i(v_i - \dot{u}_i) d\Omega + \int_{\Gamma_s} g_i(v_i - \dot{u}_i) d\Gamma. \tag{10}$$

Recall (4) for the Coulomb's friction model, $|\boldsymbol{\sigma}_T| = |\boldsymbol{F}_T|$ is considered a function of the normal contact force, $\sigma_N = F_N$. The frictional force is in the opposite direction of the tangential velocity. Therefore

$$\sigma_{Ti}\dot{u}_{Ti} = \begin{cases} 0 \quad \text{if } |\boldsymbol{\sigma}_T| < \mu|\sigma_N| \\ -\lambda|\boldsymbol{\sigma}_T|^2 = -|\boldsymbol{\sigma}_T||\dot{\boldsymbol{u}}_T| \quad \text{if } |\boldsymbol{\sigma}_T| = \mu|\sigma_N| \end{cases} \quad \text{on } \Gamma_c. \tag{11}$$

The negative sign in (11) indicates the dissipative nature of the work done by the friction. A functional to represent the virtual power of friction is introduced in [3], with the normal contact force σ_N as a functional of displacement,

$$j(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_c} \mu|\sigma_N(\boldsymbol{u})||\boldsymbol{v}_T| d\Gamma. \tag{12}$$

We can verify the following with the solution u_i, for arbitrary test function v_i,

$$\mu|\sigma_N(\boldsymbol{u})|(|\boldsymbol{v}_T| - |\dot{\boldsymbol{u}}_T|) + \sigma_{Ti}(v_{Ti} - \dot{u}_{Ti}) \geqslant 0. \tag{13}$$