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A finite element procedure for rigorous numerical enclosures on the limit load in the analysis of multibody structures

L. Angela Mihai [a], Mark Ainsworth [b,*]

[a] Mathematical Institute, University of Oxford, St Giles' 24-29, Oxford OX1 3LB, UK
[b] Department of Mathematics, University of Strathclyde, 26 Richmond Street, Glasgow G1 1XH, UK

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ABSTRACT

A rigorous finite element numerical procedure is proposed for the computation of guaranteed lower and upper bounds for the limit load of failure in a system of linear-elastic blocks in mutual non-penetrative contact with given friction. First the static and kinematic principles are formulated as continuous optimization problems and existence of a solution to the corresponding limit load problems at the infinite dimensional level is established. Two numerical approaches are devised, one for each limit load problem, to obtain actual numerical bounds on the unique critical load. The first approach uses the static limit load problem involving stresses in conjunction with a non-standard conforming finite element method to obtain a linear program from which one can derive a lower (safe) bound for the limit load and an expression for the corresponding stress field. The second approach uses the kinematic limit load problem to obtain a linear optimization problem from which one can determine an upper (unsafe) bound for the limit load and an expression for the failure mode. Together, these procedures give rise to rigorous numerical enclosures on the limit load.

1. Introduction

A challenging problem in the analysis of solids, with many important practical applications, such as the design and analysis of mechanical structures, is the prediction of the failure load and failure mode of structural assemblies. For structures assembled from deformable blocks in mutual non-penetrative contact with friction, when the compressive stresses are relatively low compared to the peak compressive stress of the blocks, failure through fracture or slip at the block-interfaces can be studied under the general principles of elasto-plastic limit load analysis (see Christiansen [10], Section 6.4). On the one hand, the problem is to identify the *values* of a superimposed load which the structure can carry without losing its equilibrium. This is the *static principle* of limit load analysis. On the other hand, one is interested in predicting the *state* of the structure under the *ultimate load*, i.e. the load that would cause dislocations that lead to loss of equilibrium. This is the *kinematic principle* of limit load analysis. For structures with prescribed or Tresca friction, the static and kinematic limit loads coincide, due to the duality principle, provided that the extrema are computed *exactly*. In practice, however, this is rarely possible. Nevertheless, existing approaches tend to concentrate on one or the other of the limit load principles to obtain an approximate value for the theoretical limit load.

In the case of a rigid, perfectly plastic solid, the problem of limit load analysis is to find the maximum static load distribution which the solid can sustain without collapsing, i.e. without plastic flow (permanent deformation) occurring, and to determine the fields of stresses and flow (velocities) in the material at the moment of collapse. For this problem, the collapse load is given by the solution of an infinite dimensional saddle-point problem which can be discretised by mixed finite elements and the resulting mathematical programming problem solved to approximate both the collapse fields of stresses and velocities. Efficient procedures for the computation of strict lower and upper bounds for the exact limit load for plane stress and plane strain models are proposed in Ciria et al. [12] and Muñoz et al. [35].

For the block assemblies, which are to be considered here, the fact that these assemblies are usually strong in compression, but not in tension, has given rise to models with various contact laws. In Drucker [13], the principle of virtual work is employed to find analytical lower and upper (albeit wide) bounds for the collapse load of elasto-plastic structures when yield is associated with Coulomb friction. As a result, the lower bound is provided by the limit load for frictionless contact, which in general is equal to zero, while the upper bound is given by the limit load for contact without relative sliding at interfaces. Geometric lower and upper bounds in the plastic analysis of structures assembled from perfectly rigid

* Corresponding author.
E-mail addresses: Mihai@maths.ox.ac.uk (L.A. Mihai), M.Ainsworth@strath.ac.uk (M. Ainsworth).

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blocks under hinging, i.e. without relative sliding of the blocks, are proposed in Heyman [23]: for a given structure under a superimposed live load, if the three requirements of equilibrium, yield, and plastic deformation are all satisfied, then the live load is equal to the unique collapse load; if only the conditions of equilibrium and yield are considered, then a safe estimate is achieved; if only the plastic mode of deformation is assumed, then an over-estimate of the true collapse load is obtained (see also [24,25]). Numerical approaches for structures assembled from rigid blocks in frictional contact described as discrete mathematical programs were pioneered by Livesley [31], and later developed by Melbourne and Gilbert [33], Fishwick [17], Baggio and Trovalusci [7], Fishwick et al. [18], Ferris and Tin-Loi [16], Orduña and Lorenço [37], Trovalusci and Baggio [38], Gilbert et al. [20]. An extension of the results for the plastic case to a finite element model of no-tension structures assembled from isotropic linear-elastic blocks is given in Maier and Nappi [32], where limit load bounds are determined by proportionally increasing the live load until a solution to the mathematical program for static equilibrium can no longer be obtained. A more general model with friction is proposed in Boothby and Brown [8], where a lower bound for the exact limit load is defined as a load for which the system is known to be stable, and an upper bound is defined as a load for which the system is known to be unstable.

While the study of finite dimensional structural models is motivated by the increasing need for effective computational techniques required in practical applications, little attention has been paid to the associated infinite dimensional problems. In this paper, planar systems of linear-elastic blocks in mutual non-penetrative contact with Tresca friction are considered, under the assumption that plastic failure in a system may occur solely through potential fracture or slip at the contact zone, while the blocks maintain their linearly elastic behaviour (see also [2–4,34]). For reviews of computational methods for contact problems in solid mechanics, we refer to Johnson [29], Wriggers [39], Acary and Brogliato [1]. By working in the theoretical framework of *variational inequalities* (Duvaut and Lions [14], Glowinski et al. [21], Hlaváček et al. [27], Kikuchi and Oden [30], Haslinger et al. [22], Eck et al. [15]), the limit load principles are formulated as variational problems, and necessary and sufficient conditions for the existence of a solution are derived. A unified theoretical treatment of the static and kinematic limit load problems is achieved by exploiting the duality principle, and two complementary numerical procedures for the evaluation of guaranteed lower and upper bounds on the limit load are devised. Specifically, the static limit load problem is formulated in terms of the stress (dual) variable and the kinematic limit load problem in terms of the displacement (primal) variable. The existence of a solution to each of these two problems is guaranteed under the following two conditions: (i) the range of admissible fields must be a non-empty set, and (ii) the associated energy functional must be coercive on the convex set of admissible fields. In the case of the static problem, if condition (i) can be shown to be satisfied, then it is easy to verify that condition (ii) also holds. On the other hand, for the kinematic problem, while condition (i) can be easily verified, it is more difficult to establish conditions on the load under which condition (ii) holds. In the first computational approach, in order to obtain a lower bound for the true limit load, the static limit load problem is discretised using the lowest order finite element method due to Arnold and Winther [6]. This offers a procedure for verifying that condition (i) holds, by exhibiting a non-empty subset of feasible finite element solutions, and in addition provides a useful numerical approximation for the static limit load problem. Whilst the use of finite elements to approximate the stress in a dual energy formulation is not unusual, their use in identifying whether or not a set is non-empty is rather novel. The resulting discrete problem reduces to a linear program in the divergence-free space, from which the lower (safe) bound for the limit load and the associated stress distribution are obtained. In the second approach, the kinematic limit load problem is expressed as a linear optimization problem from which an upper (unsafe) bound for the limit load and the failure mode are determined.

The rest of this paper is organised as follows: in Section 2, the block structure modelled as a contact problem is described, and the corresponding equivalent variational formulations are presented. In Section 3, the static limit load problem is formulated and approximated by the lowest order Arnold–Winther finite element method, to show existence of a solution and to derive computational lower bounds for the limit load. To handle the contact conditions, the degrees of freedom are modified and an equivalent formulation of the original finite element method is used. The details of the modified version of the finite element method are given in Appendix. In Section 4, the kinematic limit load problem is analysed and upper bounds for the limit load and the failure mode are derived. In Section 5, the main results are summarised and the duality principle is formally proved. In order to illustrate the theoretical results, in Section 6, several block assemblies are treated numerically.

2. Problem description

Let $\Omega = \Omega_1 \cup \cdots \cup \Omega_S \subset \mathbb{R}^2, S \geqslant 2$, represent the structure under consideration, with $\Omega_s, s = 1, \ldots, S$, polygonal linearly elastic blocks in non-penetrative frictional contact. The global boundary $\Gamma = \partial\Omega_1 \cup \cdots \cup \partial\Omega_S$ is partitioned as $\Gamma = \Gamma_C \cup \Gamma_E$, where $\Gamma_C \neq \emptyset$ is the potential contact zone consisting of the interfaces between blocks, and $\Gamma_E = \Gamma \setminus \Gamma_C$ is the exterior boundary of the overall structure.

On the contact zone Γ_C:

- the non-penetrative contact model is given by the conditions:

  $[u_N] \leqslant 0, \quad \sigma_N \leqslant 0, \quad [u_N]\sigma_N = 0,$

- the Tresca friction is defined by the relations:

  $|\sigma_T| \leqslant G, \quad [u_T](|\sigma_T| - G) = 0, \quad [u_T]\sigma_T \leqslant 0,$

where the indices N and T indicate the normal and tangential directions, respectively, which are given an arbitrarily unique value on every common edge between two blocks, u_N and u_T are the normal and tangential displacements, respectively, σ_N and σ_T are the normal and tangential stresses, respectively, $G \geqslant 0$ is the Tresca friction bound, and $[\cdot]$ represents the jump across a potential contact edge.

The exterior boundary is partitioned as $\Gamma_E = \Gamma_D \cup \Gamma_S \cup \Gamma_B$, where:

- on Γ_D a fixed support is assumed:

  $u_N = 0 \quad \text{and} \quad u_T = 0,$

- on Γ_S the conditions are of simple support:

  $u_N = 0 \quad \text{and} \quad \sigma_T = 0,$

- on $\Gamma_B \neq \emptyset$ boundary tractions are acting:

  $\sigma_N = F_N^B \quad \text{and} \quad \sigma_T = F_T^B.$

The structure is subjected to a *dead load* induced by the volume force $\mathbf{f}^D$ over Ω and the boundary tractions $\mathbf{F}^B$ on $\emptyset \neq \Gamma_B \subset \Gamma_E$.

2.1. Primal (displacement) formulation

The closed convex cone of kinematically admissible displacements is defined as follows: