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The geometric element transformation method for tetrahedral mesh smoothing

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ABSTRACT

The geometric element transformation method (GETMe) has been introduced as a new element driven approach to mesh smoothing. It is based on simple geometric transformations, which, if applied iteratively, lead to the regularization of mesh elements. Global mesh smoothing can be accomplished by successively improving the worst elements or by averaging node positions obtained by the simultaneous transformation of all elements. GETMe smoothing has been successfully applied in the case of surface meshes. As shown in this paper, this approach also naturally extends to tetrahedral mesh smoothing without major conceptual modifications. A regularizing transformation for tetrahedra is presented and a combined approach of simultaneous and sequential GETMe smoothing is described. First numerical examples yield high quality meshes superior to those obtained by other geometry-based methods. In fact, the presented results are in a majority of cases at least comparable to those obtained by a state of the art global optimization-based method.

Μηδείς ἀγεωμέτρητος εἰσίτω…
Let no one ignorant of geometry enter.
Plato

1. Introduction

In many finite element applications unstructured tessellations of the geometry under consideration play a fundamental role. Therefore, the generation of quality meshes is an essential step of the simulation process, since mesh quality has an impact on solution accuracy and the efficiency of the computational methods involved [1,2].

In the case of tetrahedral meshes, a lot of effort has been put into the development of quality mesh generators [3], which are based, for example, on advancing front techniques [4,5], Octtree methods [6], constrained Delaunay tessellations [7], and its variants [8]. In addition, specialized algorithms exist, for example, in order to efficiently remesh feature models [9]. Within this context, mesh improvement methods play an important role, since in many cases they are incorporated into the mesh generation process or used as a post-processing step. Additionally, they are also applied after mesh modifications caused, for example, by mesh merging or boundary movements.

Mesh improvement techniques can roughly be classified into methods that modify mesh topology and those which do not. Topology modifying methods are based on inserting and deleting nodes or changing connectivity by edge or face swapping (e.g. [10–12]). In contrast, topology-preserving methods, known as smoothing methods, are only based on node movements.

One of the most popular smoothing methods is Laplacian smoothing [13] due to its computational and implementational simplicity. In this, each node is successively replaced by the average of its directly connected neighboring nodes. Since this geometry-based approach is not geared towards improving element quality, the overall mesh may deteriorate and inverted elements may occur. Therefore, smart variants have been proposed, which accomplish a node update only if it leads to an improvement with respect to a given quality criterion [14]. Compared to triangular meshes, Laplacian smoothing is less efficient in the case of tetrahedral meshes, since the variety of adverse topological and geometrical configurations leading Laplacian smoothing to fail increases in 3D.

In contrast to the geometry-based approach of Laplacian smoothing, optimization-based smoothing methods determine new node positions by minimizing an objective function based on a suitable quality criterion. In order to compute new node positions, local optimization methods assess the quality of adjacent elements [15]. This leads to methods which can be combined with classical Laplacian smoothing in order to moderate the higher computational complexity [14] or even be used to untangle

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meshes [16]. The good quality achieved by local optimization-based methods can be further improved by using global optimization-based methods, which incorporate all mesh elements into the objective function [17,18]. Naturally, this leads to a higher implementational and computational complexity. Hence, very large meshes pose a problem for global optimization-based methods. This can be circumvented by the use of streaming techniques [19].

Not least in the context of optimization-based methods, algebraic quality metrics [20] have gained importance. For example, the inverse mean ratio quality metric, which measures the distance from a given element to an ideal reference element. By analyzing two energy functions based on conformal and isoparametric mappings, this metric has been recently shown in [21] to be equivalent to the angle-preserving energy. Hence, mesh optimization results in minimizing the energy. Also alternative approaches exist, which are for example based on space mapping techniques [22].

The geometric element transformation method (GETMe) introduced in [23] for triangular surface meshes and in [24] for mixed surface meshes is a geometry-based mesh smoothing method. In contrast to other smoothing methods an element driven approach is used. The key to GETMe smoothing lies within a simple geometric element transformation, which, if applied iteratively, leads to a successive regularization of the element, thus to an improvement of element quality. Thereby, the regularization speed can be controlled by specific transformation parameters, which also enables a quality driven smoothing control. First examples showed that the GETMe approach leads to superior results if compared to other geometry-based smoothing methods. Furthermore, it reaches or even outperforms the mesh quality obtained by global optimization-based smoothing methods.

Due to its general approach, GETMe smoothing naturally extends to tetrahedral meshes, as will be shown in this paper. By mainly focusing on the concepts and principles of this new approach, a regularizing transformation for tetrahedral elements according to [25] is presented in Section 2 and basic properties of the transformation are discussed. Additionally, descriptions of two basic GETMe smoothing methods using a sequential and a simultaneous element transformation approach respectively are given in Section 3. These two concepts will be combined in order to derive a new GETMe variant representing the central result of this paper. First numerical results, given in Section 4, substantiate the potential of GETMe smoothing by comparing its results to those of smart Laplacian smoothing and a global optimization-based method.

2. Transformation of single elements

The key to the geometric element transformation method lies within the proper choice of a regularizing element transformation. By this, regularizing means that if the transformation is applied iteratively to a single element, it becomes regular and hence of better quality with respect to standard element quality metrics. Consequently, this section focuses on the properties of the transformation applied to a single tetrahedron detached from the mesh smoothing context examined in subsequent sections.

2.1. Transformation of a tetrahedron

Let $T := (p_1, p_2, p_3, p_4)^t$ denote a tetrahedron with the four pairwise disjoint nodes $p_i \in \mathbb{R}^3,\ i \in \{1, \ldots, 4\}$, which is positively oriented. That is $\det(D(T)) > 0$ with

$$D(T) := (p_2 - p_1, p_3 - p_1, p_4 - p_1) \tag{1}$$

representing the (3×3)-matrix of the difference vectors, which span the tetrahedron T. Furthermore, let

$$n_1 := (p_4 - p_2) \times (p_3 - p_2),$$
$$n_2 := (p_4 - p_3) \times (p_1 - p_3),$$
$$n_3 := (p_2 - p_4) \times (p_1 - p_4),$$
$$n_4 := (p_2 - p_1) \times (p_3 - p_1)$$

denote the inside oriented face normals of T.

A new tetrahedron T' with nodes p_i' is derived from T by constructing on each node p_i the opposing face normal n_i scaled by $\sigma/\sqrt{|n_i|}$, where $\sigma \in \mathbb{R}_0^+$. That is

$$T' = \begin{pmatrix} p_1' \\ p_2' \\ p_3' \\ p_4' \end{pmatrix} := \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} + \sigma \begin{pmatrix} \frac{1}{\sqrt{|n_1|}} n_1 \\ \frac{1}{\sqrt{|n_2|}} n_2 \\ \frac{1}{\sqrt{|n_3|}} n_3 \\ \frac{1}{\sqrt{|n_4|}} n_4 \end{pmatrix}. \tag{2}$$

An initial tetrahedron T and its transformed counterpart T' using $\sigma = 1$ are depicted exemplarily in Fig. 1. In this, associated faces and normals are marked by the same color. The edges of the resulting tetrahedron T' are indicated by thick black lines.

2.2. Properties of the transformation

Due to the orientation of the normals, the transformation enlarges the tetrahedron. Thereby, the magnification is scalable by the factor σ. In the case of $\sigma = 0$ it holds that $T = T'$. Furthermore, since the normals n_i are scaled by $1/\sqrt{|n_i|}$, the transformation is scale invariant, i.e. for $s > 0$ it holds that $(sT)' = sT'$.

Of particular importance for the mesh smoothing application described later on is the regularizing effect of the transformation. That is, if the transformation is applied iteratively, the resulting tetrahedra become more and more regular. In order to assess the regularity of a tetrahedron T numerically, the mean ratio quality criterion [18,20] will be used. It is given by

$$q(T) := \frac{3 \det(S)^{2/3}}{\|S\|_F^2}, \tag{3}$$

with $\|S\|_F := \sqrt{\mathrm{trace}(S^t S)}$ denoting the Frobenius norm of the matrix $S := D(T)W^{-1}$. Hereby, $D(T)$ represents the difference matrix given by (1) and

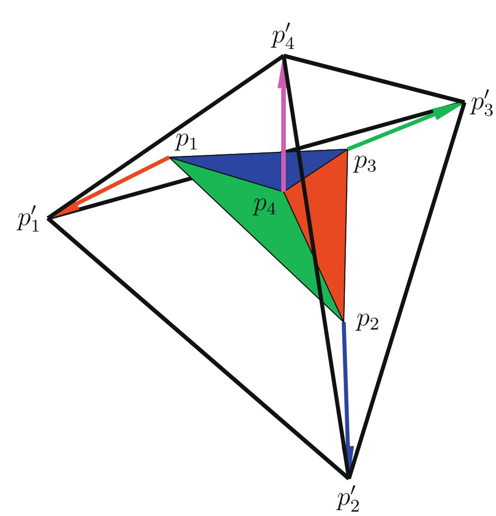

Fig. 1. Transformation of a tetrahedron using $\sigma = 1$.