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Isogeometric shell analysis with Kirchhoff–Love elements

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ABSTRACT

A Kirchhoff–Love shell element is developed on the basis of the isogeometric approach [16]. NURBS as basis functions for analysis have proven to be very efficient and offer the great feature of exact geometric representation. For a Kirchhoff–Love shell element they additionally have the significant advantage that the necessary continuities between elements are easily achieved. The element is formulated geometrically nonlinear. It is discretized by displacement degrees of freedom only. Aspects related to rotational degrees of freedom are handled by the displacement control variables, too. A NURBS-based CAD program is used to model shell structures built up from NURBS and isogeometric analysis is performed on the same model without meshing. Different examples show the performance of this method and its applicability for the integration of design and analysis.

1. Introduction

In classical shell theory one has to distinguish between thick shells ($R/t < 20$) and thin shells ($20 \leqslant R/t$). The appropriate theory to describe thick shells is the Reissner–Mindlin theory where transverse shear deformations are taken into account. For thin shells the Kirchhoff–Love theory is applicable which assumes that transverse shear deformations are negligible. For very thin shells ($1000 < R/t$) the deformations usually cannot be described by geometrically linear behaviour and a geometrically nonlinear description is necessary. Very thin shells, like thin metal sheets, play an important role in many industrial applications, e.g. in automotive and aerospace industry. Although most shell structures in practical engineering applications are in the range of thin and very thin shells and most analytical solutions for shells are based on the Kirchhoff–Love theory, the Reissner–Mindlin theory is more spread in finite element codes. This is mainly due to the fact that for Reissner–Mindlin elements only C^0 continuity is required between elements which allows the use of very simple shape functions. However, these low order elements exhibit various locking phenomena and great endeavour has been devoted in the past to circumvent these locking effects. For Kirchhoff–Love elements C^1 continuity is required between elements which is quite difficult to achieve for free-form geometries when using standard polynomials as basis functions. NURBS (nonuniform rational B-Splines) are smooth, higher order functions which are used for geometric design and have become standard in CAD (computer aided design) programs. They allow great geometric flexibility and high order continuities at the same time. NURBS are therefore ideally suited as basis functions for Kirchhoff–Love shell elements.

The idea of using NURBS as basis functions for analysis was introduced by Hughes et al. [16], and was named isogeometric analysis. In isogeometric analysis the functions from the geometry description are used as basis functions for the analysis. Thus, the analysis works on a geometrically exact model and no meshing is necessary. This offers a possibility to close the existing gap between design and analysis as both use the same geometry model. As CAD design models of thin-walled structures are often constructed by surfaces rather than volumes, shell analysis appears to be the corresponding analysis method for these applications.

In this work we present a Kirchhoff–Love shell element based on the isogeometric concept. A geometrically nonlinear formulation makes it also applicable to very thin shells under large rotations. First, a brief review of NURBS and isogeometric analysis is given. Then the newly-proposed element formulation is derived and various examples show the good performance of this element. In the last chapter the applicability of this method for the integration of CAD and analysis is demonstrated.

2. NURBS and isogeometric analysis

NURBS are a generalization of B-Splines and most of the features of NURBS also apply to B-Splines, so first a short introduction to B-Splines is given.

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2.1. B-Splines

A B-Spline is a non-interpolating, piecewise polynomial curve. It is defined by a set of control points $\mathbf{P}_i,\ i = 1,\dots,n$, the polynomial degree p and a so called knot vector $\Xi = [\xi_1, \xi_2, \dots, \xi_{n+p+1}]$. The knot vector is a set of parametric coordinates ξ_i in non-descending order which divide the B-Spline into sections. If all knots are equally spaced, the knot vector is called uniform. A B-Spline basis function is C^∞ continuous inside a knot span, i.e. between two distinct knots, and C^{p-1} continuous at a single knot. A knot value can appear more than one time and is then called a multiple knot. At a knot of multiplicity k the continuity is C^{p-k}.

If the first and the last knot have the multiplicity $p+1$, the knot vector is called open [10,21]. In a B-Spline with an open knot vector the first and the last control point are interpolated and the curve is tangential to the control polygon at the start and the end of the curve. Open knot vectors are standard in CAD applications and are assumed for the remainder of this text.

2.2. Basis functions

B-Spline basis functions are computed by the Cox–deBoor recursion formula [10,21]. It starts for $p = 0$ with:

$$N_{i,0}(\xi) = \begin{cases} 0 & \xi_i \leqslant \xi < \xi_{i+1}, \\ 1 & \text{otherwise}. \end{cases} \tag{1}$$

For $p \geqslant 1$ it is

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi). \tag{2}$$

From this formulation some important properties of B-Spline basis functions can be deduced:

- Local support, i.e. a basis function $N_{i,p}(\xi)$ is non-zero only in the interval $[\xi_i, \xi_{i+p+1}]$.
- Partition of unity, i.e. $\sum_{i=1}^{n} N_{i,p}(\xi) = 1$.
- Non-negativity, i.e. $N_{i,p}(\xi) \geqslant 0$.
- Linear independence, i.e. $\sum_{i=1}^{n} \alpha_i N_{i,p}(\xi) = 0 \iff \alpha_j = 0,\ j = 1, 2, \dots, n$.

Fig. 1 shows an example of cubic B-Spline basis functions with an open knot vector.

2.3. B-Spline curves

A B-Spline curve of degree p is computed by the linear combination of control points and the respective basis functions:

$$\mathbf{C}(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi)\mathbf{P}_i. \tag{3}$$

Fig. 2 shows an example of a cubic B-Spline with an open knot vector. Due to the open knot vector the first and last control point (P_1 and P_7) are interpolated and it can be seen that the curve is tangential to the control polygon at its start and end.

2.4. B-Spline surfaces

A B-Spline surface is computed by the tensor product of B-Spline basis functions in two parametric dimensions ξ and η. It is defined by a net of $n \times m$ control points, two knot vectors Ξ and H, two polynomial degrees p and q (which do not need to be equal), and correspondingly the basis functions $N_{i,p}(\xi)$ and $M_{j,q}(\eta)$. It is described as:

$$\mathbf{S}(\xi,\eta) = \sum_{i=1}^{n}\sum_{j=1}^{m} N_{i,p}(\xi) M_{j,q}(\eta) \mathbf{P}_{i,j}. \tag{4}$$

Fig. 3 shows an example of a quadratic B-Spline surface and its control net. Due to the open knot vectors the control points at the vertices of the surface are interpolated. The black lines on the surface mark the knots which divide the surface into elements.

2.5. NURBS

NURBS are nonuniform rational B-Splines. For NURBS each control point has additionally to its coordinates an individual weight w_i. Such a point $\mathbf{P}_i(x_i, y_i, z_i, w_i)$ can be represented with homogeneous coordinates $\mathbf{P}_i^w(w_i x_i, w_i y_i, w_i z_i, w_i)$ in a projective $\mathbb{R}^4$ space. A NURBS curve is the projection of a B-Spline in $\mathbb{R}^4$ with homogeneous control points onto $\mathbb{R}^3$ [21]:

$$\mathbf{C}(\xi) = \frac{\sum_{i=1}^{n} N_{i,p}(\xi) w_i \mathbf{P}_i}{\sum_{i=1}^{n} N_{i,p}(\xi) w_i}. \tag{5}$$

A NURBS surface is defined as

$$\mathbf{S}(\xi,\eta) = \frac{\sum_{i=1}^{n}\sum_{j=1}^{m} N_{i,p}(\xi) M_{j,q}(\eta) w_{i,j}}{\sum_{k=1}^{n}\sum_{l=1}^{m} N_{k,p}(\xi) M_{l,q}(\eta) w_{k,l}} \mathbf{P}_{i,j}. \tag{6}$$

NURBS are able to exactly represent some important geometric entities, like e.g. conic sections (i.e. circles, cylinders, spheres, etc.). Moreover, a B-Spline is a special case of a NURBS where all weights are equal and is therefore automatically contained in all the subsequent derivations for NURBS-based elements.

2.6. Continuity

For parametric curves and surfaces there are two kinds of continuity, the geometric and the parametric continuity. For the

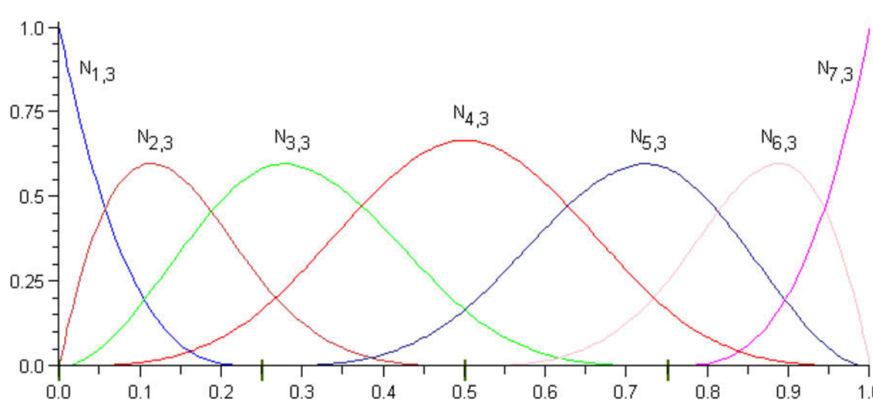

Fig. 1. Cubic B-Spline basis functions with open knot vector $\Xi = [0, 0, 0, 0, 0.25, 0.5, 0.75, 1, 1, 1, 1]$.