Comput. Methods Appl. Mech. Engrg. 198 (2009) 3278–3286

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

Collision in a cross-shaped domain – A steady 2d Navier–Stokes example demonstrating the importance of mass conservation in CFD

Alexander Linke

Weierstrass Institute for Applied Analysis and Stochastics, Berlin, Germany

ARTICLE INFO

Article history:
Received 25 February 2009
Received in revised form 29 May 2009
Accepted 8 June 2009
Available online 30 June 2009

Keywords:
Incompressible Navier–Stokes equations
Mixed finite elements
Poor mass conservation
Numerical instability

ABSTRACT

In the numerical simulation of the incompressible Navier–Stokes equations different numerical instabilities can occur. While instability in the discrete velocity due to dominant convection and instability in the discrete pressure due to a vanishing discrete Ladyzhenskaya–Babuska–Brezzi (LBB) constant are well-known, instability in the discrete velocity due to a poor mass conservation at high Reynolds numbers sometimes seems to be underestimated. At least, when using conforming Galerkin mixed finite element methods like the Taylor–Hood element, the classical grad-div stabilization for enhancing discrete mass conservation is often neglected in practical computations. Though simple academic flow problems showing the importance of mass conservation are well-known, these examples differ from practically relevant ones, since specially designed force vectors are prescribed. Therefore, we present a simple steady Navier–Stokes problem in two space dimensions at Reynolds number 1024, a colliding flow in a cross-shaped domain, where the instability of poor mass conservation is studied in detail and where no force vector is prescribed.

1. Introduction

Classical finite element analysis for mixed approximations of the incompressible Navier–Stokes equation predicts that at high Reynolds numbers special care has to be taken, in order to prevent different numerical instabilities [26]. While instability due to dominant convection is well-known for a long time [23,17,8,26] and still remains an active area of research in the finite element community [14–16,3,5–7,11,21,22,18] instability due to poor mass conservation seems to be often underestimated [24,19]. But simple academic test examples with a purpose-built force vector (e.g., rotation-free) easily show that instability due to poor mass conservation can have dramatic consequences, even at moderate Reynolds numbers [28,24,11,19,9]. Nevertheless, stabilizing poor mass conservation by the classical grad-div stabilization is not very popular in practice, since the evolving linear systems become stiff and the convergence of iterative methods like multi-grid suffers due to this stabilization operator [24].

In this paper, we present a two-dimensional steady Navier–Stokes flow in a cross-shaped domain with two inflow and two outflow channels at Reynolds number 1024. The example illustrates, under which flow conditions poor mass conservation becomes a main problem in numerical Navier–Stokes computations. The example is non-academic in the sense that there is no artificially constructed right-hand side and the flow is driven only by reasonable velocity boundary conditions. For a numerical computation with mixed finite element methods the example poses two problems: first, a large curvature of the pressure develops, due to collision of the flow in the center of the cross-shaped domain. Second, boundary layers near corner singularities evolve, since the problem is singularly perturbed.

For this flow problem, we compare the approximation quality of the classical Galerkin Taylor–Hood element $(P_2 - P_1)$ [12,4], and the divergence-free Galerkin Scott–Vogelius element $(P_2 - P_{-1})$ [33,32,30,13,2,25,6,20,19]. In order to prevent instability due to dominant convection, we resolve the boundary layers by a customized anisotropic mesh. The chosen sequence of meshes also assures the Ladyzhenskaya–Babuska–Brezzi (LBB) stability of both mixed finite element methods [13,2,25]. Then the Galerkin Taylor–Hood element delivers numerical approximations that are spoiled by spurious oscillations due to poor mass conservation, while the divergence-free Galerkin Scott–Vogelius element yields stable and accurate numerical solutions [19]. The better accuracy of the Galerkin Scott–Vogelius method is numerically demonstrated by an investigation of the convergence behavior of both methods with respect to a reference solution. This superior behavior of the Scott–Vogelius element is remarkable, since the algebraic space of discretely divergence-free functions is much larger in the case of the Taylor–Hood element than in the case of the Scott–Vogelius element.

E-mail address: linke@wias-berlin.de

0045-7825/$ - see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.cma.2009.06.016

2. The Stokes, Oseen and Navier–Stokes problems

We consider the following system of partial differential equations for $(\mathbf{u}, p)$ in a polygonal domain $\Omega \subset \mathbb{R}^2$.

$$\begin{aligned} &-\nu\Delta\mathbf{u} + (\mathbf{a}(\mathbf{u})\cdot\nabla)\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in } \Omega,\\ &\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega,\\ &\mathbf{u} = \mathbf{u}_D \quad \text{on } \Gamma_D,\\ &\mathbf{u}\cdot\mathbf{n} = 0, \quad \frac{\partial(\mathbf{u}\cdot\mathbf{t})}{\partial\mathbf{n}} = 0 \quad \text{on } \Gamma_S. \end{aligned} \tag{1}$$

The boundary $\partial\Omega$ is split in two different parts $\partial\Omega = \Gamma_D \cup \Gamma_S$ with $\Gamma_D \cap \Gamma_S$ being zero-dimensional. On Γ_D Dirichlet boundary conditions are prescribed, while on Γ_S symmetry boundary conditions apply. We assume that $\mathbf{u}_D \in [C(\Gamma_D)]^2$ is continuous and can be continued to a function $\mathbf{u}_B \in [H^1(\Omega)]^2$. The continuation can be constructed, e.g., by solving the following problem: find $\mathbf{u}_B \in \mathbf{A}$, with the affine trial space

$$\mathbf{A} := \{\mathbf{v} \in [H^1]^2 : \text{trace}_{\Gamma_D}(\mathbf{v}) = \mathbf{u}_D \wedge \text{trace}_{\Gamma_S}(\mathbf{v})\cdot\mathbf{n} = 0\}$$

and solve

$$(\nabla\mathbf{u}_B, \nabla\mathbf{v}) = 0$$

for all $\mathbf{v} \in \mathbf{V}$ with

$$\mathbf{V} := \{\mathbf{v} \in [H^1]^2 : \text{trace}_{\Gamma_D}(\mathbf{v}) = \mathbf{0} \wedge \text{trace}_{\Gamma_S}(\mathbf{v})\cdot\mathbf{n} = 0\}.$$

Here, $(\cdot,\cdot)$ denotes the L^2-scalar product. Further we assume that $\nu > 0$ is a constant and that $\mathbf{f} \in [L^2(\Omega)]^2$ holds.

For the convection term $(\mathbf{a}(\mathbf{u})\cdot\nabla)\mathbf{u}$ we will investigate three different choices

$$\mathbf{a}(\mathbf{u}) = \begin{cases} \mathbf{0}, & \text{the Stokes problem}\\ \mathbf{a}, & \text{the Oseen problem}\\ \mathbf{u}, & \text{the (nonlinear) Navier–Stokes problem.} \end{cases}$$

In the case of the Oseen problem, we assume that the conditions $\nabla\cdot\mathbf{a} = 0$ and $\mathbf{a}_{|\partial\Omega} = \mathbf{u}_{|\partial\Omega}$ hold and that $\mathbf{a}$ is as smooth as $\mathbf{u}$ is. Each of these equations describes the steady distribution of a velocity field $\mathbf{u}$ and a pressure field p in an incompressible fluid. The Stokes model is applied, when inertial forces are negligible and only frictional forces are important. The Navier–Stokes model is applied, when both frictional and inertial forces are relevant. The Oseen model has a rather limited physical meaning. It is a linearized Navier–Stokes problem and often serves as a model problem for a numerical analysis of the full Navier–Stokes problem.

For a weak formulation of problem (1), we introduce the Sobolev space

$$Q := L^2_0(\Omega) = \left\{ q \in L^2(\Omega) : \int_\Omega q(x)dx = 0 \right\}$$

and the new variable $\mathbf{u}_{\text{hom}} := \mathbf{u} - \mathbf{u}_B$. Obviously, for $\mathbf{u}_{\text{hom}}$ apply homogenous Dirichlet boundary conditions on Γ_D.

The weak formulation of this problem can be stated in the following saddle point form: find $(\mathbf{u}_{\text{hom}}, p) \in \mathbf{V} \times Q =: \mathbf{X}$ such that

$$\begin{aligned} &a(\mathbf{u}_{\text{hom}}, p, \mathbf{v}_{\text{hom}}, q) + b(\mathbf{u}_{\text{hom}}, p, \mathbf{v}_{\text{hom}}, q) + b(\mathbf{v}_{\text{hom}}, q, \mathbf{u}_{\text{hom}}, p)\\ &\quad = l(\mathbf{v}_{\text{hom}}, q) \end{aligned} \tag{2}$$

for all $(\mathbf{v}_{\text{hom}}, q) \in \mathbf{X}$. Here, the forms $a(\cdot,\cdot) : \mathbf{X}\times\mathbf{X} \to \mathbb{R}, b(\cdot,\cdot) : \mathbf{X}\times\mathbf{X} \to \mathbb{R}$, and $l : \mathbf{X} \to \mathbb{R}$ are defined as

$$\begin{aligned} a(\mathbf{u}_{\text{hom}}, p, \mathbf{v}_{\text{hom}}, q) &:= \nu(\nabla\mathbf{u}_{\text{hom}}, \nabla\mathbf{v}_{\text{hom}}) \\ &\quad + ((\mathbf{a}(\mathbf{u}_B + \mathbf{u}_{\text{hom}})\cdot\nabla)(\mathbf{u}_B + \mathbf{u}_{\text{hom}}), \mathbf{v}_{\text{hom}}), \end{aligned} \tag{3}$$

$$\begin{aligned} b(\mathbf{u}_{\text{hom}}, p, \mathbf{v}_{\text{hom}}, q) &:= -(\nabla\cdot\mathbf{u}_{\text{hom}}, q), \\ l(\mathbf{v}_{\text{hom}}, q) &:= (\mathbf{f}, \mathbf{v}_{\text{hom}}) - \nu(\nabla\mathbf{u}_B, \nabla\mathbf{v}_{\text{hom}}) + (\nabla\cdot\mathbf{u}_B, q) \end{aligned} \tag{4}$$

for all $(\mathbf{u}_{\text{hom}}, p), (\mathbf{v}_{\text{hom}}, q) \in \mathbf{X}$. The form $b(\cdot,\cdot)$ is bilinear and bounded, $l(\cdot)$ is linear and bounded, and $a(\cdot,\cdot)$ is linear in the second argument.

In the linear Stokes and Oseen cases the problem can be simplified further. In the Stokes problem the term $\mathbf{a}(\mathbf{u})$ drops out, and the form $a(\cdot,\cdot)$ is actually bilinear and bounded. In the Oseen case by moving one term to the right-hand side, we introduce the slightly modified forms

$$\begin{aligned} a_{\text{Oseen}}(\mathbf{u}_{\text{hom}}, p, \mathbf{v}_{\text{hom}}, q) &:= \nu(\nabla\mathbf{u}_{\text{hom}}, \nabla\mathbf{v}_{\text{hom}}) + ((\mathbf{a}\cdot\nabla)\mathbf{u}_{\text{hom}}, \mathbf{v}_{\text{hom}}), \\ l_{\text{Oseen}}(\mathbf{v}_{\text{hom}}, q) &:= (\mathbf{f}, \mathbf{v}_{\text{hom}}) - \nu(\nabla\mathbf{u}_B, \nabla\mathbf{v}_{\text{hom}}) + (\nabla\cdot\mathbf{u}_B, q) \\ &\quad - ((\mathbf{a}\cdot\nabla)\mathbf{u}_B, \mathbf{v}_{\text{hom}}) \end{aligned} \tag{5}$$

for all $(\mathbf{u}_{\text{hom}}, p), (\mathbf{v}_{\text{hom}}, q) \in \mathbf{X}$, and we must solve the problem: *find* $(\mathbf{u}_{\text{hom}}, p) \in \mathbf{X}$ *such that*

$$\begin{aligned} &a_{\text{Oseen}}(\mathbf{u}_{\text{hom}}, p, \mathbf{v}_{\text{hom}}, q) + b(\mathbf{u}_{\text{hom}}, p, \mathbf{v}_{\text{hom}}, q) + b(\mathbf{v}_{\text{hom}}, q, \mathbf{u}_{\text{hom}}, p)\\ &\quad = l_{\text{Oseen}}(\mathbf{v}_{\text{hom}}, q) \end{aligned} \tag{6}$$

holds for all $(\mathbf{v}_{\text{hom}}, q) \in \mathbf{X}$. The form $a_{\text{Oseen}}(\cdot,\cdot)$ is bilinear and bounded. Then an existence and uniqueness theory for the Stokes and Oseen problem is straight-forward. We define the space of divergence-free, weakly differentiable vector functions

$$\mathbf{V}_0 := \{\mathbf{v} \in \mathbf{V} : (\nabla\cdot\mathbf{v}, q) = 0 \quad \text{for all } q \in Q\}, \tag{7}$$

and the bilinear forms $a(\cdot, 0, \cdot, 0)$ and $a_{\text{Oseen}}(\cdot, 0, \cdot, 0)$ restricted to the product space $\mathbf{V}_0 \times \mathbf{V}_0$ are coercive, due to $\nabla\cdot\mathbf{a} = 0$. The existence of the pressure p is guaranteed by the Ladyzhenskaja condition, i.e., on the considered domain Ω it holds that for all $q \in Q$ there is a velocity $\mathbf{v} \in \mathbf{V}$ with $\nabla\cdot\mathbf{v} = q$ such that

$$\|\nabla\mathbf{v}\|_0 \leqslant C\|p\|_0,$$

with a constant C only depending on the shape of Ω, see [12].

The existence theory for the steady Navier–Stokes problem is more involved and needs the application of the theory of pseudo-monotone operators, see Ref. [27]. Then uniqueness can be expected a priori only for large values of ν, i.e., $\nu = \mathcal{O}(1)$.

Below we will present a two-dimensional Navier–Stokes problem with $\mathbf{f} \equiv \mathbf{0}$, demonstrating the importance of mass conservation in numerical approximations of the Navier–Stokes equation. Then the flow is driven only by the inhomogeneous Dirichlet boundary conditions, and the rotation-free part of the convection term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ arises as a source of a numerical instability. This numerical instability will be illustrated by theoretical considerations concerning an appropriate Stokes model problem with homogeneous Dirichlet boundary conditions and non-zero right-hand side $\mathbf{f}$.

3. Conforming Galerkin mixed finite elements

A conforming Galerkin mixed finite element discretization for the incompressible Stokes, Oseen or Navier–Stokes equations, starts directly from the weak formulation in Eq. (2). Applying this weak formulation, we choose finite-dimensional function spaces $\mathbf{V}^h \subset \mathbf{V}$ and $Q^h \subset Q$ serving as trial and test functions for the weak formulation in Eq. (2). Here, the term *Galerkin* means that we use the same function spaces for trial and test functions, while the term *conforming* emphasizes that the discrete spaces $\mathbf{V}^h$ and Q^h are really subspaces of $\mathbf{V}$ and Q. Since the mathematical nature of the quantities velocity and pressure in the incompressible Navier–Stokes equation are quite different, the term *mixed* is applied.

For the discretization of the incompressible Stokes, Oseen and Navier–Stokes equations, we use the classical Taylor–Hood element and the Scott–Vogelius element. Therefore, let $\mathcal{T}^h$ denote a triangulation of the domain Ω without hanging nodes. For each triangle $\overline{T} \in \mathcal{T}^h$, we define