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Hybrid-Trefftz stress and displacement elements for axisymmetric incompressible biphasic media

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ABSTRACT

The stress and displacement models of the hybrid-Trefftz finite element formulation are applied to the solution of incompressible axisymmetric saturated porous media. The use of a Trefftz approximation basis ensures that all domain conditions of the problem are satisfied in a strong form, namely the equilibrium, the constitutive and the strain–displacement relations and the mixture incompressibility condition. The alternative stress and displacement models are fully complementary in terms of approximation criteria. The stress (displacement) model is derived from the direct approximation of the stress and pressure fields (the displacements in the solid and fluid phases) in the domain of the element. The displacements of the solid phase and the normal displacement of the fluid phase are approximated independently on the boundary of the stress element and used to enforce in a weak form the inter-element and boundary equilibrium conditions on the forces in the solid phase and on the fluid pressure. The boundary approximation used in the displacement element is on the solid surface forces and the fluid pressure, and is used to enforce on average the inter-element and boundary displacement continuity conditions. The resulting finite element governing systems are sparse, well-suited to adaptive refinement and parallel processing, and their coefficients are defined by boundary integral expressions. The energy statements associated with the formulation are recovered and sufficient conditions for the uniqueness of the finite element solutions are stated. Benchmark tests on hydrated soft tissue modelling are used to assess the performance of the alternative stress and displacement models of the hybrid-Trefftz formulation.

1. Introduction

This paper closes the report on a study on the modelling of the response of hydrated soft tissues with hybrid-Trefftz finite elements [1,2]. It extends to axisymmetric problems the formulation and implementation of the elements developed for the analysis of two-dimensional problems [3,5,6], and is used to establish a direct comparative analysis of the alternative stress and displacement models.

The parabolic model proposed in Mow et al. [7] for incompressible biphasic media is adopted here. This model is first discretized in the time dimension in a format that can accommodate alternative time integration procedures, namely the trapezoidal rules frequently used in the finite element modelling of the response of soft tissues, as illustrated in the work reported by Spilker and his co-workers, e.g., [8] and the spectral decomposition method that is used here to assess the response of the hybrid-Trefftz elements in both frequency and time domains [9].

The resulting boundary value problem is discretized next using a technique in every aspect similar to that applied in the development of hybrid and hybrid-Trefftz elements for two-dimensional problems. Thus, and to avoid unnecessary duplication, the finite element equations for the alternative displacement and stress models of the hybrid formulation are stated and their specific roles are recalled. The hybrid-Trefftz variants are obtained by constraining the approximation bases to the solution set of the governing system of differential equations, the distinguishing feature of the Trefftz approach [10].

The first part of the paper closes with the presentation of the finite element governing systems. The main aspects of its numerical implementation are briefly recalled. The associated energy statements and the sufficient conditions that ensure the uniqueness of the finite element solutions presented for two-dimensional problems are here adapted to axisymmetric problems.

The second part of the paper addresses the assessment of the performance of the hybrid-Trefftz elements when applied to the frequency and time domain analyses. The results that are presented show that the Trefftz elements for hydrated soft tissue modelling preserve the high performance features that have been consistently reported on their application to progressively wider classes of modelling problems since the pioneering work of Jirousek and Leon [11,12].

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The spectral analysis tests are used to illustrate the convergence rates and patterns obtained with p- and h-refinement procedures and to analyse the sensitivity of the elements to gross shape distortion and to quasi-incompressibility conditions set on each phase of the mixture. The forcing frequencies used in the spectral analysis tests are selected to range a wide variation in the relation between the typical dimension of the element and the wavelength of the excitation.

A non-periodic spectral decomposition time integration procedure, implemented on a high-order wavelet time approximation basis, is used to obtain the results obtained for the time domain analysis tests. It is shown that the coupling of the combination of the Trefftz approximation in space (implemented on relatively coarse meshes) with the wavelet approximation in time (applied in a single time increment) produces adequate estimates for the stress, pressure and displacement fields, at both local and global levels and at every instant of the loading process.

2. Boundary value problem

After implementation of the time integration method (see Appendix A), the equilibrium, compatibility and elasticity conditions in the domain V of a saturated porous element are written as follows, where ω is the forcing (or algorithmic) frequency, i is the imaginary unit and subscripts s and f identify quantities associated with the solid and fluid phases of the mixture:

$$\begin{bmatrix} \mathbf{D} & \phi_s\nabla \\ \boldsymbol{O} & \phi_f\nabla \end{bmatrix}\begin{Bmatrix} \boldsymbol{\sigma}_s \\ p \end{Bmatrix} + \begin{Bmatrix} \boldsymbol{b}_s + \boldsymbol{b}_o \\ \boldsymbol{b}_f - \boldsymbol{b}_o \end{Bmatrix} = i\omega\zeta\begin{Bmatrix} \boldsymbol{u}_s - \boldsymbol{u}_f \\ \boldsymbol{u}_f - \boldsymbol{u}_s \end{Bmatrix} \quad \text{in } V, \tag{1}$$

$$\begin{Bmatrix} \boldsymbol{\varepsilon}_s \\ \gamma = 0 \end{Bmatrix} = \begin{bmatrix} \mathbf{D}^* & \boldsymbol{O} \\ \phi_s\nabla^* & \phi_f\nabla^* \end{bmatrix}\begin{Bmatrix} \boldsymbol{u}_s \\ \boldsymbol{u}_f \end{Bmatrix} \quad \text{in } V, \tag{2}$$

$$\boldsymbol{\sigma}_s = \boldsymbol{k}\boldsymbol{\varepsilon}_s \quad \text{in } V. \tag{3}$$

Vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ define the independent components of the stress and strain tensors, p and γ are the pressure and the volumetric change of the mixture, and $\boldsymbol{b}$ and $\boldsymbol{u}$ are the body force and displacement vectors, respectively. Vector $\boldsymbol{b}_o$ is used to define the equivalent body forces associated with the initial condition of the problem, and parameters ζ, ϕ_s and ϕ_f define the diffusive drag and the solid and fluid fraction ratios, with $\phi_s + \phi_f = 1$. The (linear) differential operators, the divergence matrix $\mathbf{D}$ and the gradient vector ∇, and their conjugates $\mathbf{D}^*$ and ∇^*, are defined in Appendix B for axisymmetric problems, as well as the stiffness matrix, $\boldsymbol{k}$.

Four complementary regions are identified on the boundary of the element, Γ, to define the following force and displacement continuity conditions:

$$\boldsymbol{t} = \boldsymbol{N}\boldsymbol{\sigma}_s + \phi_s\boldsymbol{n}p = \bar{\boldsymbol{t}} \quad \text{on } \Gamma_t, \tag{4}$$

$$\phi_f p = \bar{p} \quad \text{on } \Gamma_p, \tag{5}$$

$$\boldsymbol{u}_s = \bar{\boldsymbol{u}} \quad \text{on } \Gamma_u, \tag{6}$$

$$\boldsymbol{n}^t\boldsymbol{u}_f = \bar{w} \quad \text{on } \Gamma_w. \tag{7}$$

In the Neumann boundary conditions (4) and (5), $\bar{p}$ is the pressure prescribed on the fluid and matrix $\boldsymbol{N}$ collects the components of the unit outward normal vector, $\boldsymbol{n}$, that establish the equilibrium conditions on stress and prescribed force, $\bar{\boldsymbol{t}}$, components. In the Dirichlet conditions (6) and (7), vector $\bar{\boldsymbol{u}}$ defines the displacements prescribed on the solid matrix and $\bar{w}$ is the outward normal component of the displacement in the fluid.

Eqs. (4)–(7), which can be written to account for mixed boundary conditions, hold for inter-element continuity conditions, in which case the prescribed term defines the displacements and/or the forces applied to the interfaces between connecting elements. In addition, the elasticity condition (3) can be extended to include creep and stress relaxation terms.

3. Governing differential equation

Combination of Eqs. (1)–(3) leads to the mixed (Navier–Beltrami) system of equations,

$$\begin{cases} k_p^{-2}\nabla(\nabla^*\boldsymbol{u}_s) + k_s^{-2}\widetilde{\nabla}(\widetilde{\nabla}^*\boldsymbol{u}_s) + \phi_f(\boldsymbol{u}_s - \boldsymbol{u}_f) = -ik^{-2}(\boldsymbol{b}_s + \boldsymbol{b}_f), \\ \nabla p = ik^2\phi_f(\boldsymbol{u}_f - \boldsymbol{u}_s) - \phi_f^{-1}(\boldsymbol{b}_f - \boldsymbol{b}_o), \\ \nabla^*(\phi_s\boldsymbol{u}_s + \phi_f\boldsymbol{u}_f) = 0, \end{cases} \tag{8}$$

where $\widetilde{\nabla}$ is the anti-gradient vector and $\widetilde{\nabla}^*$ its conjugate: $\nabla^*\widetilde{\nabla}(\circ) = \widetilde{\nabla}^*\nabla(\circ) = 0$. They are defined in Appendix B, where the P- and S-wavenumbers k_p and k_s are given in terms of parameter $k^2 = \zeta\omega\phi_f^{-2}$.

The homogeneous system (8) has three sets of solutions, namely constant pressure (frozen) modes, harmonic pressure modes and Helmholtz pressure modes. They are defined as follows,

$$\nabla p = \boldsymbol{0}, \quad \boldsymbol{u}_s = \nabla\varphi \quad \text{and} \quad \boldsymbol{u}_f = \boldsymbol{u}_s, \\ \text{with } \nabla^2\varphi = 0, \tag{9}$$

$$p = \varphi, \quad \boldsymbol{u}_s = \widetilde{\nabla}\Psi \quad \text{and} \quad \boldsymbol{u}_f \\ = \boldsymbol{u}_s - ik^{-2}\phi_f^{-1}\nabla\varphi, \quad \text{with } \widetilde{\nabla}(\widetilde{\nabla}^2\Psi) = -ik^{-2}k_s^2\nabla\varphi, \tag{10}$$

$$p = \phi, \quad \boldsymbol{u}_s = ik^{-2}\nabla\phi \quad \text{and} \quad \boldsymbol{u}_f \\ = -\phi_s\phi_f^{-1}\boldsymbol{u}_s, \quad \text{with } \nabla^2\phi + k_p^2\phi = 0, \tag{11}$$

where $\nabla^2(\circ)=\nabla^*\nabla(\circ)$ is the Laplacian and $\widetilde{\nabla}^2(\circ) = \widetilde{\nabla}^*\widetilde{\nabla}(\circ)$ its conjugate. The potentials are defined in cylindrical and spherical co-ordinate systems in Appendices C and D, respectively [1].

4. Formulations and models

The finite element formulation used here to solve problem (8) under boundary conditions (4)–(7) is termed *hybrid* because it involves independent approximations in the domain and on the boundary of the element, and *Trefftz* to acknowledge the implementation of the method proposed by Trefftz [10] to solve boundary value problems as an alternative to the method proposed earlier by Ritz [13].

Instead of constraining the approximation basis to satisfy in a strong form the essential boundary conditions and determining the weights of the basis by enforcing in weak form the remaining conditions of the problem, the Trefftz method generalizes the classical, analytical method to solve boundary value problems: the approximation basis is defined as a linear combination of the solutions of the governing differential equation (8), meaning that all domain conditions (1)–(3) are satisfied in strong form (the so-called Trefftz constraint), and the weights of the basis are determined enforcing the essential boundary conditions in weak form.

Different models are possible for the resulting Ritz or Trefftz formulation, depending on the choice of the essential boundary conditions. Two models are derived and tested here, depending on whether the essential boundary conditions are identified with the Neumann conditions (4) and (5), as in the *stress* model, or with the Dirichlet conditions (6) and (7), as in the *displacement* model. A consequence of this option is that the stress (displacement) model may produce, under certain conditions, statically (kinematically) admissible solutions, that is, solutions that satisfy in strong form the domain and boundary equilibrium (compatibility) conditions of the problem.

In order to control the enforcement of the continuity conditions strictly in terms of either forces or displacements, the domain approximation bases involve fields of the same nature in each model, namely static variables (stresses) in the stress model and kinematic variables (displacements) in the displacement element.