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Explicit solutions for a long-wave model with constant vorticity

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ABSTRACT

Explicit parametric solutions are found for a nonlinear long-wave model describing steady surface waves propagating on an inviscid fluid of finite depth in the presence of a linear shear current. The exact solutions, along with an explicit parametric form of the pressure and streamfunction give a complete description of the shape of the free surface and the flow in the bulk of the fluid. The explicit solutions are compared to numerical approximations previously given in Ali and Kalisch (2013), and to numerical approximations of solutions of the full Euler equations in the same situation Teles da Silva and Peregrine (1988). These comparisons show that the long-wave model yields a fairly accurate approximation of the surface profile as given by the Euler equations up to moderate waveheights. The fluid pressure and the flow underneath the surface are also investigated, and it is found that the long-wave model admits critical layer recirculating flow and non-monotone pressure profiles similar to the flow features of the solutions of the full Euler equations.

1. Introduction

Background vorticity can have a significant effect on the properties of waves at the surface of a fluid [1–10]. In particular, in the seminal paper of Teles da Silva and Peregrine [11], it was found that the combination of strong background vorticity and large amplitude leads to a number of unusual wave shapes, such as narrow and peaked waves and overhanging bulbous waves. In the present contribution, we continue the study of a simplified model equation which admits some of the features found in [11]. The equation, which has its origins in early work of Benjamin [12], has the form

$$\left(Q+\frac{\omega_0}{2}u^2\right)^2\left(\frac{du}{dx}\right)^2 = -3\left(\frac{\omega_0^2}{12}u^4+gu^3-(2R-\omega_0 Q)u^2+2Su-Q^2\right), \qquad (1)$$

where we denote the volume flux per unit span by Q, the momentum flux per unit span and unit density corrected for pressure force by S, and the energy density per unit span by R. The gravitational acceleration is g and the constant vorticity is $-\omega_0$. The total flow depth as measured from the free surface to the rigid bottom is given by the function $u(x)$.

Eq. (1) was recently studied in [13]. It was found that solutions of this equation exhibit similar properties as solutions of the full Euler equations displayed in [11]. In particular, in [13] an expression for the pressure was developed, and it was shown that the pressure may become non-monotone in the case of strong background vorticity. Indeed, it was shown in [13] that if $|\omega_0|$ is big enough, the maximum fluid pressure at the bed is not located under the wavecrest. Such behavior is usually only found in transient problems (cf. [14]). Moreover in some cases, the pressure near the crest of the wave may be below atmospheric pressure. In contrast, in an irrotational flow beneath a traveling surface water wave, the pressure is monotone with depth, and no overhanging profiles are possible [15,16].

The purpose of the present work is two-fold. First, we develop a method by which Eq. (1) can be solved *exactly*. The resulting solutions are compared to the numerical approximations found in [13] and to some of the solutions of the full Euler equations from [11]. Secondly, more features of the solutions of (1) are discussed. Using a similar analysis as in [13], the streamfunction is constructed, and it is found that solutions of (1) may feature recirculating flow and pressure inversion. These features may have an impact on the study of sediment resuspension. Indeed, while it is generally accepted that the main mechanism for sediment resuspension is turbulence due to flow separation in the presence of strong viscous shear stresses [17–19], the strongly

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non-monotone pressure profiles exhibited by the solutions of (1) may represent a more fundamental mechanism for particle suspension than the viscous theory. In particular, in Fig. 13 it can be seen that very strong shear allows for near atmospheric pressure close to the bed, and there are regions of high pressure situated below regions of lower pressure leading to an upwardly directed force in the fluid.

The geometric setup of the problem is explained as follows. Consider a background shear flow $U_0 = \omega_0 z$, where ω can be positive or negative (cf. Fig. 1). Superimposed on this background flow is wave motion at the surface of the fluid. As observed by a number of authors [12,11,9], a linear shear current can be taken as a first approximation of more realistic shear flows with more complex structures.

If it is assumed that the free surface describes a steady periodic oscillatory pattern, then the flow underneath the free surface can be uniquely determined [20–22]. In the presence of vorticity, uniqueness holds under certain conditions, but in some cases, there is loss of uniqueness, and this allows the existence of critical layers in the fluid [23]. For the purpose of studying periodic traveling waves, one may use a reference frame moving with the wave. This change of reference frame leads to a stationary problem in the fundamental domain of one wavelength. The incompressibility guarantees the existence of the streamfunction ψ and if constant vorticity $\omega = -\omega_0$ is stipulated, the streamfunction satisfies the Poisson equation

$$\Delta\psi = \psi_{xx} + \psi_{zz} = \omega_0, \quad \text{in } 0 < z < \eta(x). \tag{2}$$

As explained in [24,25], the three parameters Q, S and R are defined as follows. If $\psi = 0$ on the streamline along the flat bottom, then Q denotes the total volume flux per unit width given by

$$Q = \int_0^{\eta} \psi_z dz. \tag{3}$$

Thus Q is the value of the streamfunction ψ at the free surface. The flow force per unit width S is defined by

$$S = \int_0^{\eta} \left\{ \frac{P}{\rho} + \psi_z^2 \right\} dz, \tag{4}$$

and the energy per unit mass is given by

$$R = \frac{1}{2}\psi_z^2 + \frac{1}{2}\psi_x^2 + g\eta \quad \text{on } z = \eta(x). \tag{5}$$

Finally, the pressure can be expressed as

$$P = \rho\Big(R - gz - \frac{1}{2}(\psi_x^2 + \psi_z^2) + \omega_0\psi - \omega_0 Q\Big). \tag{6}$$

It is well known that the quantities Q and S do not depend on the value of x [24]. Using the fact that S is a constant, the derivation of the model equation (1) can be effected by assuming that the waves are long, scaling z by the undisturbed depth h_0, x by a typical wavelength L, and expanding in the small parameter $\beta = h_0^2/L^2$. This yields (1) as an approximate model equation describing the shape of the free surface. In order to distinguish from the free surface η in the full Euler description, we call the unknown of Eq. (1) u which is an approximation of η. The derivation of (1) was given in [24], where it was shown that (1) is expected to be valid as an approximate model equation describing waves on the surface of the shear flow if the wavelength is long compared to the undisturbed depth of the fluid. On the other hand, a detailed analysis of the derivation explained in [24] shows that there are no assumptions on the amplitude of the waves. Thus at least formally, the model (1) can be expected to describe waves of large amplitude.

2. Explicit solutions

In order to obtain solutions of (1) given in explicit form, we apply the change of variables

$$\frac{dy}{ds} = \frac{du}{dx}\left(Q + \frac{\omega_0}{2}u^2\right), \qquad y(s) = u(x).$$

This gives us a new equation for $y(s)$ in the form

$$\left(\frac{dy}{ds}\right)^2 = -3\left(\frac{\omega_0^2}{12}y^4 + gy^3 - (2R - \omega_0 Q)y^2 + 2Sy - Q^2\right), \tag{7}$$

and the relation

$$\frac{ds}{dx} = \frac{1}{Q + y^2\omega_0/2}. \tag{8}$$

Integrating (8) we have

$$x(s) = \int^s \left(Q + \frac{\omega_0}{2}y^2\right) d\xi - x_1 \tag{9}$$

where x_1 is a constant of integration, written explicitly for convenience. We want to solve (7) for $y(s)$ and plug our solution into (9). We notice that in the variables y and $\frac{dy}{ds}$ the equation describes an elliptic curve of genus one [26]. Hermite's Theorem [27, p. 394] states that for a uniform solution to exist we need $\int ds$ to be an abelian integral of the first kind. This condition is indeed satisfied and we proceed with using a birational transformation to put (7) in the standard Weierstraß form

$$\left(\frac{dy_0}{dx_0}\right)^2 = 4y_0^3 - g_2 y_0 - g_3, \tag{10}$$

where the transformation is given in Box I, and g_2 and g_3 are the lattice invariants

$$g_2 = -768QR\omega_0 + 768R^2 - 1152Sg,$$
$$g_3 = 2048Q^3\omega_0^3 - 6144Q^2R\omega_0^2 - 6912Q^2g^2 + 6144QR^2\omega_0 - 4608QSg\omega_0 + 2034S^2\omega_0^2 - 4096R^3 + 9216RSg.$$

It is well known that the solution to (10) is $y_0(x_0) = \wp(x_0 + c_0; g_2, g_3)$, where $\wp$ is the Weierstraß P function and c_0 is an arbitrary constant [28,26]. We invert the birational transformation to determine the exact solution to (7) as

$$y(s) = \frac{A + B\wp'((s+c_0)/4; g_2, g_3) + C\wp((s+c_0)/4; g_2, g_3)}{\wp^2((s+c_0)/4; g_2, g_3) + D\wp((s+c_0)/4; g_2, g_3) + E},$$

with

$$A = -288Q^2g - 96Q\omega_0 S + 192RS, \qquad B = \sqrt{12}Q,$$
$$C = -24S,$$
$$D = 8Q\omega_0 - 16R, \qquad E = 64Q^2\omega_0^2 - 64QR\omega_0 + 64R^2.$$

This gives $u(x(s))$ in the form

$$u(x(s)) = \frac{A + B\wp'((s+c_0)/4; g_2, g_3) + C\wp((s+c_0)/4; g_2, g_3)}{\wp^2((s+c_0)/4; g_2, g_3) + D\wp((s+c_0)/4; g_2, g_3) + E}, \tag{12}$$

as a function of the parameter s. If we express $x(s)$ as a function of s, then we have a parametric representation for $u(x)$, the surface elevation. From (9) we have

$$x(s) = Qs - x_1 + \frac{\omega_0}{2}\int^s y^2(\xi)d\xi. \tag{13}$$

Expanding and simplifying $y(s)^2$ gives

$$y^2 = \frac{4B\wp^3 + C^2\wp^2 + (2AC - B^2g_2)\wp + (A^2 - B^2g_3)}{(\wp^2 + D\wp + E)^2} + \frac{2AB - 2BC\wp}{(\wp^2 + D\wp + E)^2}\wp', \tag{14}$$